A SCALAR-TENSOR THEORY OF GRAVITATION COMPATIBLE WITH MACH'S PRINCIPLE

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A SCALAR-TENSOR THEORY OF GRAVITATION

COMPATIBLE WITH MACH'S PRINCIPLE

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GLOSSARY OF ABBREVIATIONS

MP	Mach's Principle
GR	General Relativity
ST	scalar-tensor
DPEM	direct particle equations of motion
FSSMD	finite spherically symmetric mass distribution
LHS	left hand side
RHS	right hand side
RS	red shift
MRS	Machian red shift

UNITS AND NOTATION

Greek indices will run from 1 to 4 rather than 0 to 3 as some authors prefer. The line element, ds, will have the unit of length. The signature is chosen to be S = -2. Thus the Minkowski line element is

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2 \; .$$

The time coordinate, x^4, is defined as $x^4 = ct$. Since x^4 is not taken to be imaginary, the Minkowski (or Galilean) metric tensor is

$$\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{pmatrix}$$

and this array is denoted by $\eta_{\mu\nu}$. Natural units (in which c = 1, G = 1) will not be used. The energy, tensor, $T_{\mu\nu}$, will have the units of mass per unit volume.

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SUMMARY

This thesis consists basically of the analysis of a Mach's Principle compatible scalar-tensor theory of gravitation. The paper begins by discussing Mach's Principle in detail and showing that we can adopt either of two fundamentally different viewpoints as to its meaning, depending on whether we postulate that space-time structure depends completely or only partially on the matter content of the universe. These two viewpoints are called, respectively, the strong and weak forms of Mach's Principle. It is then shown that General Relativity, while compatible with the weak form, cannot be made compatible with the strong form. In order to obtain a theory which is compatible with the strong form we are led to a particular set of scalar-tensor field equations.

We then proceed to analyze this scalar-tensor theory in detail. Initially we show that it contains General Relativity as a special case. Then we prove that the scalar-tensor theory does have the required Mach's Principle compatibility. In doing so we obtain the solution of the field equations in the region outside a finite spherically symmetric mass distribution. The metric turns out to be conformally Schwarzschildian (i.e. $g_{\mu\nu} = h\, g'_{\mu\nu}$ where $g'_{\mu\nu}$ is the Schwarzschild metric), a fact which is of great importance in being to handle the theory mathematically. We also obtain the direct particle equations of motion (the equations of motion of free particles as determined by the field equations themselves), which turn out not to be the metrical geodesics.

The direct particle equations of motion are then calculated for the conformally Schwarzschildian metric mentioned above. The difference between these equations and the general relativistic result is shown to be negligible for small regions (cosmologically speaking) like the solar system. Thus the usual three predictions of General Relativity still obtain in the more complicated scalar-tensor theory.

We then make a somewhat detailed investigation into the question as to whether there might be other theories (scalar-tensor or tensor) which are compatible with the strong form of Mach's Principle as we have defined it.

Finally we look at the consequences of the new theory - the foremost of which is the complete relativity of clocks, a result which might initially seem quite surprising. However, it is not really so, for the requirement of compatibility with the strong form of Mach's Principle results in the complete dependence of space-time structure on the presence of matter. As the rate of a clock naturally depends on the space-time structure in its vicinity, the clock rate must in turn depend on the existence and distribution of matter in the universe. In particular, clocks alone in an otherwise empty universe are found not to run at all (in sharp contrast to their behavior in General Relativity). However, as matter is introduced into the region near a clock, its rate is drastically affected.

It is shown that this clock relativity leads to the prediction of a large red shift for light approaching a galaxy from a suitably chosen companion body. We then argue that it is possible to interpret quasars as simply the companion bodies of cosmologically local galaxies. Lastly we examine some of the present day quasar data and show that it is quite compatible with this analysis. And moreover, since quasars are thus allowed to be "brought in" to cosmologically local distances, where no unusual (such as gravitational collapse) energy sources need be postulated, we consider that there is definite merit in the scalar-tensor theory.

PART I

THE NEED FOR A SCALAR-TENSOR THEORY OF GRAVITATION

CHAPTER I

MACH'S PRINCIPLE AND GENERAL RELATIVITY

It has long been known that Einstein's equations of the gravitational field are in some sense incompatible with Mach's Principle. The reason it is necessary to say "some sense" is that there is no general agreement as to the physical or mathematical meaning of Mach's Principle. As Louis Witten (1) puts it, "Most physicists will say something to the effect that Mach's Principle means that the inertia of an object depends on the existence and distribution of other objects in the universe. Each individual physicist then takes off from this in a direction of his own ..."

Since in relativistic theories a body's inertia is determined by the metric at the object's location, Witten's statement may be reformulated as follows: Mach's Principle (hereafter denoted MP) implies that space-time itself depends on the existence and distribution of the matter in the universe. But here a fundamental question arises. For by "depends" does one mean "depends completely" or "depends partially"? We will refer to these viewpoints, respectively, as the strong and weak forms of MP. In the first part of this thesis we will show that General Relativity (hereafter denoted GR), while compatible with the weak form, cannot be made compatible with the strong form. Thus if we desire to retain the strong form we are forced to abandon, or at least modify, the Einstein field equations.

In searching for a new set of field equations, we will be guided by postulating a precise, unambiguous, mathematical statement of the strong form of MP and requiring that the new theory be compatible with this statement.

Einstein (2) himself believed that a theory compatible with MP (strong or weak form) should make the following three predictions:

1) The inertia of a body must increase when ponderable masses are piled up in its neighborhood.
2) A body must experience an accelerating force when neighboring masses are accelerated, the force being in the same direction as that of the accelerating masses.
3) A rotating hollow body must generate in its interior both Coriolis and centrifugal force fields.

All three of these effects are indeed predicted to a certain extent by GR. However, the calculations are based mainly on two assumptions These are firstly that the fields are sufficiently weak so as to differ only slightly from a background Minkowski space and secondly that the contributions of the different sources can be summed linearly. However, the second assumption is in general false since the Einstein field equations are non-linear. And the first assumption is inherently contradictory to the strong form of MP because if one lets the already weak fields become vanishingly weak, the background Minkowski space still remains.

So at best these three results of GR can only be considered as support for the weak MP. In fact Graves (3) writes

> Einstein seems content with the...weaker version of the principle. It at least states that space-time is not absolute in the sense that Newton's was, sitting in aloof splendor and totally unaffected by the matter moving around in it. But unless the strong version is true, space at least retains some character of its own and some degree of independence from matter.

Graves goes on to say

> A strong form of MP would demand that there should be no limit for the metric tensor in the case of infinitely weak fields. If matter were to disappear, so would space-time itself also be expected to disappear. The question becomes whether matter simply modifies an already existing space-time structure, or whether it is the sole source of that structure.

In conclusion Graves states that "Einstein at least supported the strong version as a hope or ultimate goal even if he could only prove the weak one compatible with GR."

The idea expressed by Graves that as matter disappears so does space-time is an equivalent expression of the strong form of MP. For if space-time depends completely on the existence of the matter in the universe then it must surely disappear (in the sense of being structureless) if the matter were made to vanish. We now show how to express this idea mathematically.

Consider Figure 1-1. It shows the mass M (assumed to be spherically symmetric) as the sole matter in an otherwise empty universe. This mass will certainly generate some space-time structure in its immediate vicinity, i.e. $g_{\mu\nu} \neq 0$ near M. But if we wish to satisfy the strong form of MP we must require that as M disappears, so does $g_{\mu\nu}$. That is

$$g_{\mu\nu} \rightarrow 0 \quad \text{as} \quad M \rightarrow 0 \; . \tag{1-1}$$

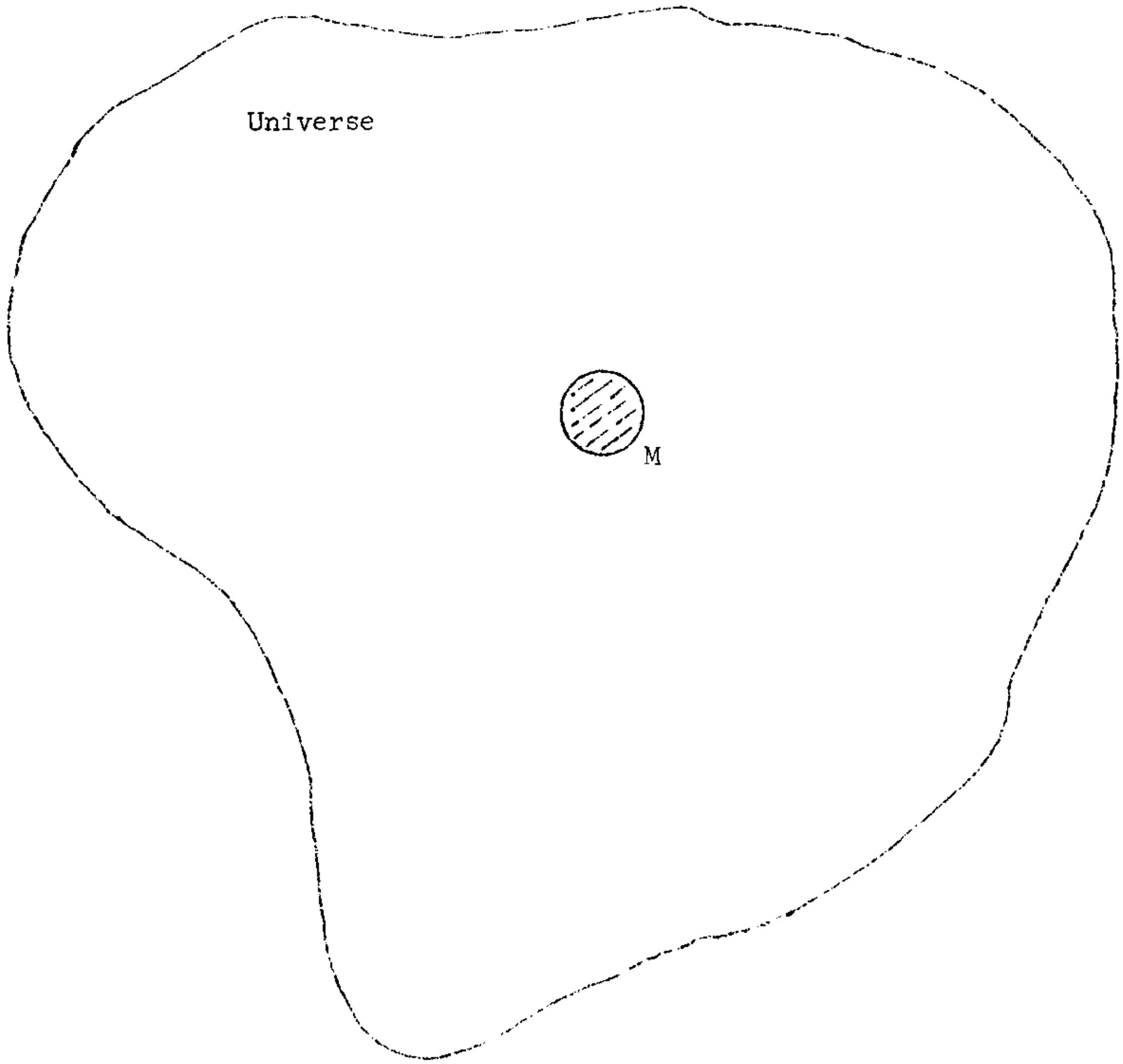

Figure 1-1. Single Mass in Otherwise Empty Universe

But this condition is not met in GR. Consider the Schwarzschild metric valid in the region exterior to the mass M, i.e.

$$ds^2 = -(1-R^*/r)^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2$$

$$+ (1-R^*/r)c^2dt^2 \tag{1-2}$$

where

$$R^* = 2GM/c^2 \ . \tag{1-3}$$

Here G is the gravitational constant and c is the speed of light. Clearly if $M \to 0$, $R^* \to 0$ and (1-2) becomes

$$ds^2 \to -dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + c^2dt^2 \ . \tag{1-4}$$

Using the standard spherical to Cartesian coordinate transformation we can write (1-4) as

$$ds^2 \to -dx^2 - dy^2 - dz^2 + c^2dt^2 \ . \tag{1-5}$$

This can also be expressed as

$$g_{\mu\nu} \to \eta_{\mu\nu} \tag{1-6}$$

where $\eta_{\mu\nu} \equiv \text{diag}\,(-1, -1, -1, +1)$ is the Minkowski (or Galilean) tensor. Thus we do not have $g_{\mu\nu} \to 0$ as $R^* \to 0$ as the strong form of MP would demand.

Furthermore there is a second major difficulty associated with the Schwarzschild line element (1-2). For as $r \to \infty$ we again get $g_{\mu\nu} \to \eta_{\mu\nu}$, a result which is also at odds with MP. For as

Silberstein (4) states*

> ...the adoption of the galilean or inertial tensor at infinity would be tantamount to giving up the requirement of the [complete] relativity of inertia. For whereas the inertia or mass of a particle generally depends upon the [$g_{\alpha\beta}$] and these are even at the surface of the sun but slightly different from [$\eta_{\alpha\beta}$], the mass of the particle at infinity would differ but very little from what it is near the sun or other celestial giants. In fine, the bulk of its mass would be independent of other bodies and if the particle existed alone in the whole universe, it would still retain practically all its mass.

Thus in order to have a complete relativity of inertia the mass of a test particle would have to vanish as it is removed infinitely far from M. The only way to effect this is to require

$$g_{\mu\nu} \to 0 \quad \text{as} \quad r \to \infty . \tag{1-7}$$

Combining this with (1-1), we see that the strong form of MP demands that

$$g_{\mu\nu} \to 0 \quad \text{as} \quad R^* \to 0 \quad \text{or} \quad r \to \infty \tag{1-8}$$

where $g_{\mu\nu}$ is the metric generated by the mass R*. We emphasize that (1-8) represents two separate conditions. That is we must have both $g_{\mu\nu} \to 0$ as $R^* \to 0$ and $g_{\mu\nu} \to 0$ as $r \to \infty$. This can be generalized to

$$g_{\mu\nu} \to 0 \quad \text{as} \quad T_{\mu\nu} \to 0 \quad \text{or} \quad r \to \infty \tag{1-9}$$

where $T_{\mu\nu}$ is the energy tensor of the given distribution of matter.

It might be argued here that some of this discussion is misleading since the Schwarzschild line element is obtained in GR by

*For definiteness Silberstein considers the mass M to be the sun.

using the boundary condition $g_{\mu\nu} \to \eta_{\mu\nu}$ as $r \to \infty$. Indeed many authors make use of this condition in their derivations. However, it appears that it is not necessary to do so. Anderson (5) states at the conclusion of his derivation "Finally we point out that asymptotically $g_{\mu\nu} \sim \eta_{\mu\nu}$ as $r \to \infty$. However, it was not necessary to require this asymptotic behavior to obtain our solution."

Thus we regard the Minkowski background to be inherent in GR and that it is this essential feature of GR that forever prevents it from being compatible with the strong form of MP, which we have taken to be expressed by (1-8).

CHAPTER II

GENERAL FORM OF NEW FIELD EQUATIONS

Having obtained a mathematical statement of MP (here and hereafter when referring to MP it is to be understood that we mean the strong form) and shown that GR is inherently incompatible with it, we now turn to the second phase of the overall problem. We must obtain a new set of gravitational field equations, the solution of which in the case of a finite spherically symmetric mass distribution (hereafter denoted FSSMD), will satisfy (1-8).

Einstein's theory can be summarized essentially by the two equations

$$R_{\alpha\beta} - \tfrac{1}{2} R\, g_{\alpha\beta} = -\varkappa\, T_{\alpha\beta} \qquad (2\text{-}1)$$

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \qquad (2\text{-}2)$$

where $\varkappa \equiv 8\pi G/c^2$. There are, of course, in addition a defining equation for $T_{\alpha\beta}$ and the equations of motion of free particles which for the moment we will assume are given by the geodesics of the space. (The question of interdependence will be considered later.)

If we restrict ourselves to a purely tensor theory the number of reasonable modifications of this theory is very limited. This is because the Einstein tensor $G_{\alpha\beta}$ ($G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta}$) is already as general as possible if no derivatives higher than second order in $g_{\alpha\beta}$ are permitted and if the field equation is to be linear in these second derivatives. (See Chapter IX for further discussion of tensor modifications.) For this reason we were led to consider scalar-tensor (hereafter denoted ST) theories, for then the possibilities are numerous, and at the same time the concept of a generally covariant theory of gravity is easily preserved.

Denoting the scalar field by Ψ, the ST theory is assumed to be of the form

$$A_{\alpha\beta}(g_{\alpha\beta},\Psi) = -\kappa T_{\alpha\beta} \tag{2-3}$$

$$f(g_{\alpha\beta},\Psi) = 0 \tag{2-4}$$

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{2-5}$$

where (2-4) is the additional equation which must be supplied since we now have eleven unknowns, the ten $g_{\alpha\beta}$ and Ψ. Of course, $f(g_{\alpha\beta},\Psi) = 0$ must be a covariant (scalar) equation and $A_{\alpha\beta}$ must be a tensor. We will impose no particular restrictions on $A_{\alpha\beta}$ and f except that neither shall contain derivatives of $g_{\alpha\beta}$ or Ψ higher than second order. Also associated with the ST theory is a defining equation for $T_{\alpha\beta}$ and the equations of motion of free particles. Because of the interdependence in non-linear field theories between the field equations and the equations of motion, we cannot assume that free particles will move on geodesics. Thus we shall in all probability have to calculate the equations of motion directly. We will refer to these calculated equations of motion as the direct particle equations of motion (denoted

DPEM) in order to distinguish them from the geodesic equations of motion.

For the purposes of finding a ST theory which is compatible with MP as expressed by (1-8), it is profitable to write $g_{\alpha\beta}$ as

$$g_{\alpha\beta} = h(\Psi)g'_{\alpha\beta} \tag{2-6}$$

in which case (2-3), (2-4) and (2-5) become

$$B_{\alpha\beta}(g'_{\alpha\beta},\Psi) = -\varkappa T_{\alpha\beta} \tag{2-7}$$

$$k(g'_{\alpha\beta},\Psi) = 0 \tag{2-8}$$

$$ds^2 = h(\Psi)g'_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{2-9}$$

where $h(\Psi)$ is an as yet unspecified scalar function of Ψ. Then we can satisfy (1-8) provided $h(\Psi)$ can be made to vanish as $r \to \infty$ and $R^* \to 0$ for the solution in the region exterior to a FSSMD. Of course, $g'_{\alpha\beta}$ must be well behaved as the limits $r \to \infty$ and $R^* \to 0$ are taken.

In fact, ideally what we want is for $g'_{\alpha\beta}$ to be equal to the Schwarzschild metric and $h(\Psi)$ to be some slowly varying function of r and R^* which vanishes as $r \to \infty$ or $R^* \to 0$. Then the exterior FSSMD solution would be

$$ds^2 = h(\Psi)\{-(1-R^*/r)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + (1-R^*/r)c^2dt^2\} \tag{2-10}$$

with

$$h(\Psi) \to 0 \quad \text{as} \quad r \to \infty \quad \text{or} \quad R^* \to 0 \; . \tag{2-11}$$

By requiring that $h(\Psi)$ be slowly varying we ensure that its effect (as compared to that of $g'_{\alpha\beta}$) is negligible over "small" distances. We will then have for all practical purposes the Schwarzschild solution holding within the solar system. Then (provided also that the DPEM are indistinguishable from the metrical geodesics) the three familiar predictions of GR (the double bending of starlight, the gravitational red shift and the precession of Mercury's perihelion) will still obtain.

We will show in Part II that such a ST theory does exist. In Part III we will examine the consequences of this theory with particular attention to the possibility of explaining quasars as local objects whose red shifts follow naturally from the new line element for spherical symmetry.

PART II

THE PROPOSED SCALAR-TENSOR THEORY

CHAPTER III

STATEMENT OF THE SCALAR-TENSOR THEORY

The following equations form a MP compatible ST theory of the gravitational field:

$$R'^{\alpha\beta} - \tfrac{1}{2} R' g'^{\alpha\beta} = -\kappa T^{\alpha\beta} \tag{3-1}$$

$$\Box' \Psi = 0 \tag{3-2}$$

$$ds^2 = e^{1/\Psi^3} g'_{\alpha\beta} dx^\alpha dx^\beta .* \tag{3-3}$$

The significance of the primes will be discussed shortly. The quantity $T^{\alpha\beta}$ is the stress-energy tensor of the distribution of matter in the region of interest. It has precisely the same form here as it does in the Einstein theory. For instance for a perfect fluid of mass density ρ and pressure p

$$T^{\alpha\beta} = (\rho + \frac{p}{c^2}) \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} - \frac{p}{c^2} g^{\alpha\beta} . \tag{3-4}$$

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*The significance of the exponent 3 in the term e^{1/Ψ^3} is to ensure that this factor closely approximates unity over the dimensions of the solar system. Higher integer values (4,5,6,...) would also work, but lower values (viz. 1 and 2) would not. This point is discussed further in Chapter VIII and IX. Also Chapter XII consists of a proof of the statement that e^{1/Ψ^3} is negligible within the solar system.

(See, for example, Tolman (6).) $R_{\alpha\beta}$, the covariant form of $R^{\alpha\beta}$, is the Ricci tensor and is given by

$$R_{\alpha\beta} = - \left\{{\lambda \atop \alpha\beta}\right\}_{,\lambda} + \left\{{\lambda \atop \alpha\lambda}\right\}_{,\beta} + \left\{{\lambda \atop \mu\beta}\right\}\left\{{\mu \atop \alpha\lambda}\right\} - \left\{{\lambda \atop \alpha\beta}\right\}\left\{{\mu \atop \lambda\mu}\right\} \tag{3-5}$$

where

$$\left\{{\lambda \atop \alpha\beta}\right\} = g^{\lambda\mu}[\alpha\beta,\mu] \tag{3-6}$$

and

$$[\alpha\beta,\mu] = \tfrac{1}{2}(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu}) \ . \tag{3-7}$$

The quantities $[\alpha\beta,\mu]$ and $\left\{{\lambda \atop \alpha\beta}\right\}$ are called, respectively, the Christoffel symbols of the first and second kind. The inverse of $g_{\alpha\beta}$ is written as $g^{\alpha\beta}$ and satisfies

$$g^{\lambda\alpha} g_{\lambda\beta} = \delta^{\alpha}_{\beta} \tag{3-8}$$

where δ^{α}_{β} is the Kronecker delta. The comma notation is used to indicate partial differentiation, so, for example

$$g_{\alpha\beta,\mu} \equiv \frac{\partial}{\partial x^{\mu}} (g_{\alpha\beta}) \ . \tag{3-9}$$

The curvature invariant, R, is defined by

$$R = g^{\alpha\beta} R_{\alpha\beta} = g_{\alpha\beta} R^{\alpha\beta} \ . \tag{3-10}$$

The contravariant form of the Ricci tensor, $R^{\alpha\beta}$, is given by

$$R^{\alpha\beta} = g^{\alpha\lambda} g^{\beta\mu} R_{\lambda\mu} \ . \tag{3-11}$$

The constant κ is equal to $8\pi G/c^2$. The quantity $\Box\Psi$ is defined as

$$\Box\Psi = g^{\alpha\beta}\,\Psi_{;\alpha\beta} \tag{3-12}$$

where the semicolon denotes covariant differentiation. Obviously $\Box\Psi$ is a scalar. Since

$$\Psi_{;\alpha\beta} = \Psi_{,\alpha\beta} - \left\{{\lambda \atop \alpha\beta}\right\}\Psi_{,\lambda} \tag{3-13}$$

we have

$$\Box\Psi = g^{\alpha\beta}\left[\Psi_{,\alpha\beta} - \left\{{\lambda \atop \alpha\beta}\right\}\Psi_{,\lambda}\right] . \tag{3-14}$$

The significance of the primes in (3-1) and (3-2) is as follows: a primed quantity has the same value as its GR counterpart except that $g_{\alpha\beta}$ is everywhere replaced by $g'_{\alpha\beta}$. Thus for example from (3-5), (3-6) and (3-7)

$$R'_{\alpha\beta} = -\left\{{\lambda \atop \alpha\beta}\right\}'_{,\lambda} + \left\{{\lambda \atop \alpha\lambda}\right\}'_{,\beta} + \left\{{\lambda \atop \mu\beta}\right\}'\left\{{\mu \atop \alpha\lambda}\right\}' - \left\{{\lambda \atop \alpha\beta}\right\}'\left\{{\mu \atop \lambda\mu}\right\}' \tag{3-15}$$

$$\left\{{\lambda \atop \alpha\beta}\right\}' = g'^{\lambda\mu}\,[\alpha\beta,\mu]' \tag{3-16}$$

$$[\alpha\beta,\mu]' = \tfrac{1}{2}(g'_{\alpha\mu,\beta} + g'_{\beta\mu,\alpha} - g'_{\alpha\beta,\mu}) . \tag{3-17}$$

Also (3-14) gives

$$\Box'\Psi = g'^{\alpha\beta}\left[\Psi_{,\alpha\beta} - \left\{{\lambda \atop \alpha\beta}\right\}'\Psi_{,\lambda}\right] . \tag{3-18}$$

The appearance of the primes in these various equations is an unavoidable complication resulting from our desire to retain the symbol $g_{\alpha\beta}$ to represent the metric tensor. That is, we still write (3-3) as

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{3-19}$$

so that

$$g_{\alpha\beta} = e^{1/\Psi^3}\, g'_{\alpha\beta} \ . \tag{3-20}$$

Hence in the field equations as given by (3-1) and (3-2) where $g'_{\alpha\beta}$ (and not $g_{\alpha\beta}$) is being calculated, we need primed quantities.

We shall have to be very careful when raising and lowering indices since in general we should like to shift primed indices with $g'_{\alpha\beta}$ and unprimed indices with $g_{\alpha\beta}$. Thus we will not write the covariant form of (3-1) as $R'_{\alpha\beta} - \frac{1}{2}R'\, g'_{\alpha\beta} = -\varkappa\, T_{\alpha\beta}$. This is because $T^{\alpha\beta}$ is defined entirely in terms of $g_{\alpha\beta}$ (see (3-4)), and we wish to write

$$T_{\alpha\beta} = g_{\alpha\lambda}\, g_{\beta\mu}\, T^{\lambda\mu} \ . \tag{3-21}$$

To find the covariant form of (3-1) we first multiply through by $g'_{\alpha\lambda}g'_{\beta\mu}$ which gives

$$R'_{\lambda\mu} - \frac{1}{2}\, R'\, g'_{\lambda\mu} = -\, \varkappa\, g'_{\alpha\lambda}g'_{\beta\mu}\, T^{\alpha\beta} \ . \tag{3-22}$$

Then using (3-20) in the form

$$g'_{\alpha\beta} = e^{-1/\Psi^3}\, g_{\alpha\beta} \tag{3-23}$$

we obtain with the help of (3-21)

$$R'_{\lambda\mu} - \frac{1}{2} R' g'_{\lambda\mu} = - \kappa e^{-2/\Psi^3} T_{\lambda\mu} \tag{3-24}$$

which is the desired covariant form of the tensor field equation. Similarly, the mixed form of this field equation can easily be shown to be

$$R'^{\alpha}_{\beta} - \frac{1}{2} R' g'^{\alpha}_{\beta} = - \kappa e^{-1/\Psi^3} T^{\alpha}_{\beta} . \tag{3-25}$$

Finally, we note that because of the Bianchi identities we have the well known identity

$$(R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta});\beta \equiv 0 . \tag{3-26}$$

If we simply replace $g_{\alpha\beta}$ everywhere with $g'_{\alpha\beta}$ this becomes

$$(R'^{\alpha\beta} - \frac{1}{2} R' g'^{\alpha\beta})_{;'\beta} \equiv 0 \tag{3-27}$$

where the symbol ;' indicates that the coderivative is taken with respect to $g'_{\alpha\beta}$. From (3-1) we have immediately that

$$T^{\alpha\beta}{}_{;'\beta} = 0 . \tag{3-28}$$

This equation is very important since it enables us to determine the DPEM of the new theory (see Chapter VI) and in addition provides us with a conservation law. It should not be surprising that (3-28) serves in these two capacities since it is just the analogue in the ST theory of the general relativistic equation $T^{\alpha\beta}{}_{;\beta} = 0$ which performs these functions in GR.

Finally, we note that in any region of space in which the energy tensor vanishes, the field equations (3-1) and (3-2) become

$$\left.\begin{aligned} R'^{\alpha\beta} &= 0 \\ \Box'\Psi &= 0 \end{aligned}\right\} \quad . \tag{3-29}$$

CHAPTER IV

GENERAL RELATIVITY AS SPECIAL CASE

The ST theory can be made to reduce to GR by taking the scalar field Ψ to be a constant and then letting the constant become large. If $\Psi = \text{const} \equiv \Psi_o$, (3-2) is satisfied identically while (3-1) and (3-20) become

$$R'^{\alpha\beta} - \tfrac{1}{2} R' \, g'^{\alpha\beta} = - \varkappa \, T^{\alpha\beta} \tag{4-1}$$

$$g_{\alpha\beta} = e^{1/\Psi_o^3} \, g'_{\alpha\beta} \; . \tag{4-2}$$

As Ψ_o becomes large $e^{1/\Psi_o^3} \rightarrow 1$ and

$$g'_{\alpha\beta} \rightarrow g_{\alpha\beta} \; . \tag{4-3}$$

Then (4-1) becomes

$$R^{\alpha\beta} - \tfrac{1}{2} R \, g^{\alpha\beta} = - \varkappa \, T^{\alpha\beta} \tag{4-4}$$

which is the GR result.

Since the ST theory contains GR it must also contain Special Relativity and classical mechanics since both of these are contained within GR.

CHAPTER V

MACH'S PRINCIPLE COMPATIBILITY

We now undertake the task of showing that the ST theory discussed in Chapter III does indeed possess MP compatibility as discussed in Chapter I and summarized by (1-8). We wish to show that the solution of the ST field equations (3-1) and (3-2) in the case of a FSSMD has the property $ds^2 \to 0$ as $r \to \infty$ or $R^* \to 0$.

Consider then a spherically symmetric body of mass M and radius R (see Figure 5-1). We will be interested only in the exterior solution, i.e. the solution in the region $r > R$. It can be shown (7) that the most general line element exhibiting spherical symmetry has the form

$$ds^2 = -e^{\alpha(r,t)}dr^2 - e^{\beta(r,t)}(r^2d\theta^2 + r^2\sin^2\theta d\varphi^2) + e^{\gamma(r,t)}c^2dt^2 \qquad (5\text{-}1)$$

where as indicated the unknown functions α, β, γ are at most functions of r and t. If we further insist that the solution be static, this equation becomes

$$ds^2 = -e^{\alpha(r)}dr^2 - e^{\beta(r)}(r^2d\theta^2 + r^2\sin^2\theta d\varphi^2) + e^{\gamma(r)}c^2dt^2 \, . \qquad (5\text{-}2)$$

We may write this as

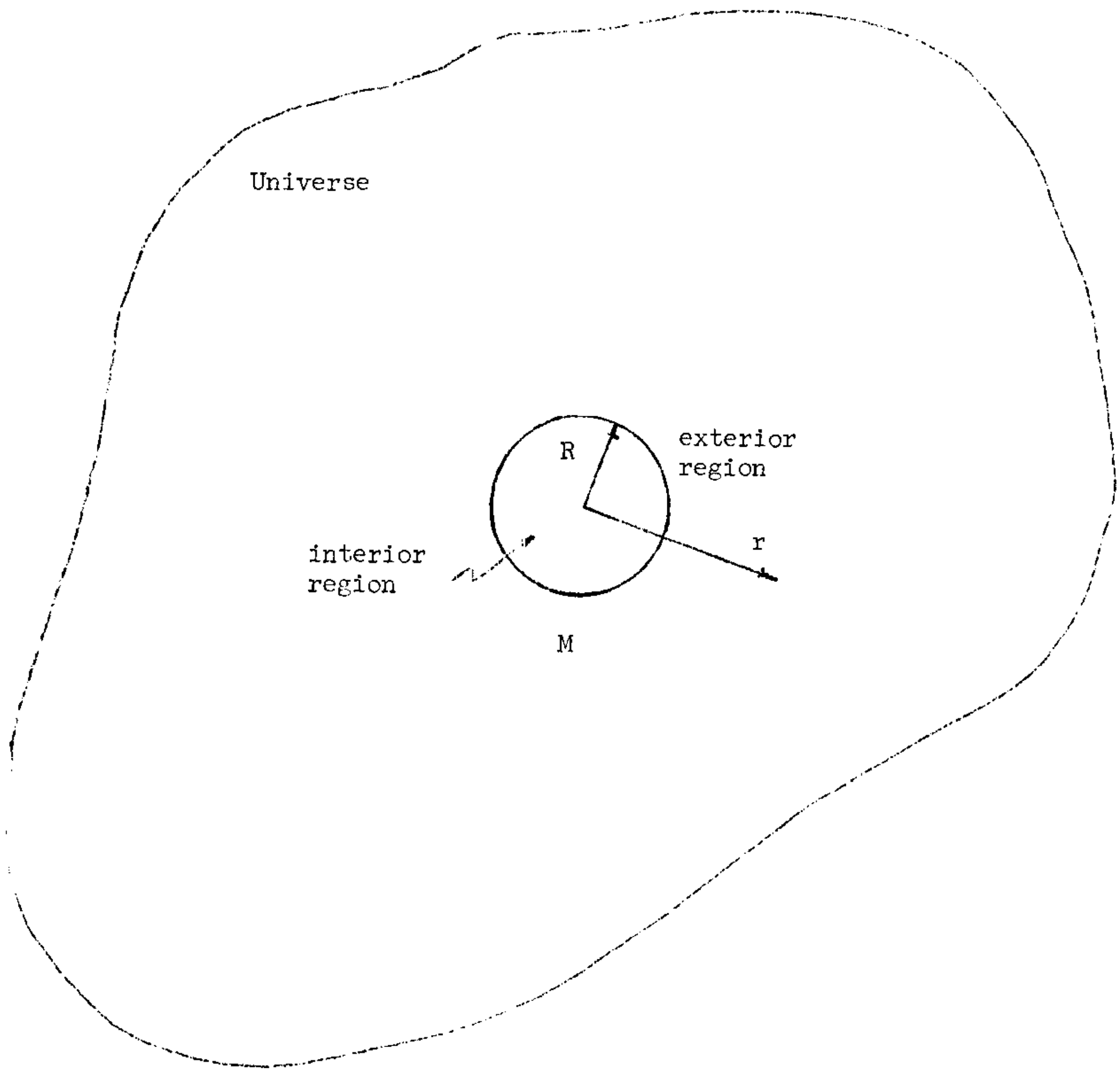

Figure 5-1. A Finite Spherically Symmetric Mass Distribution

$$ds^2 = e^{\beta}[-e^{\alpha-\beta}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + e^{\gamma-\beta}c^2dt^2] \tag{5-3}$$

or by renaming variables

$$ds^2 = e^{\gamma}[-e^{\alpha}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + e^{\beta}c^2dt^2] \; . \tag{5-4}$$

By comparison of this expression with (3-3), viz.,

$$ds^2 = e^{1/\Psi^3} \, g'_{\alpha\beta} \, dx^{\alpha} \, dx^{\beta} \tag{5-5}$$

we may evidently make the identification

$$\Psi = 1/\gamma^{1/3} \tag{5-6}$$

$$g'_{11} = -e^{\alpha} \tag{5-7}$$

$$g'_{22} = -r^2 \tag{5-8}$$

$$g'_{33} = -r^2\sin^2\theta \tag{5-9}$$

$$g'_{44} = e^{\beta} \tag{5-10}$$

$$g'_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta \; . \tag{5-11}$$

These give immediately

$$g'^{11} = -e^{-\alpha} \tag{5-12}$$

$$g'^{22} = -1/r^2 \tag{5-13}$$

$$g'^{33} = -1/r^2\sin^2\theta \tag{5-14}$$

$$g'^{44} = e^{-\beta} \tag{5-15}$$

$$g'^{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta \ . \tag{5-16}$$

Although we shall not need every one of them, we list here for the sake of completeness all of the non-vanishing Christoffel symbols of the second kind:

$$\left\{ {1 \atop 11} \right\}' = \tfrac{1}{2}\, d\alpha/dr \tag{5-17}$$

$$\left\{ {1 \atop 22} \right\}' = -\, r\, e^{-\alpha} \tag{5-18}$$

$$\left\{ {1 \atop 33} \right\}' = -\, r\, e^{-\alpha} \sin^2\theta \tag{5-19}$$

$$\left\{ {1 \atop 44} \right\}' = \tfrac{1}{2}\, e^{\beta-\alpha}\, d\beta/dr \tag{5-20}$$

$$\left\{ {2 \atop 12} \right\}' = \left\{ {2 \atop 21} \right\}' = \left\{ {3 \atop 13} \right\}' = \left\{ {3 \atop 31} \right\}' = 1/r \tag{5-21}$$

$$\left\{ {4 \atop 14} \right\}' = \left\{ {4 \atop 41} \right\}' = \tfrac{1}{2}\, d\beta/dr \tag{5-22}$$

$$\left\{ {3 \atop 23} \right\}' = \left\{ {3 \atop 32} \right\}' = \cot\theta \tag{5-23}$$

$$\left\{ {2 \atop 33} \right\}' = -\sin\theta \cos\theta \ . \tag{5-24}$$

In the region of interest, $r > R$, the energy tensor $T^{\alpha\beta}$ vanishes so (3-1) becomes

$$R'^{\alpha\beta} - \tfrac{1}{2}\, R'\, g'^{\alpha\beta} = 0 \tag{5-25}$$

or equivalently

$$R'^{\alpha\beta} = 0 \ . \tag{5-26}$$

But the solution of this equation with $g'_{\alpha\beta}$ given by (5-7) to (5-11) is well known to be

$$e^{\beta} = e^{-\alpha} = 1 - A/r \tag{5-27}$$

since these equations are mathematically identical to the Schwarzschild problem of GR. Here A is a constant of integration. A is evaluated in GR by requiring that as r becomes large compared to R, the gravitational effects of the Schwarzschild metric closely resemble those which would be calculated classically from Newton's law of gravitation. It turns out that A = R* where as usual R* is the Schwarzschild radius of the mass M ($R^* = 2GM/c^2$). We will show later (Chapter VIII) that this result is unchanged by the ST theory. Thus we have in all

$$e^{\beta} = e^{-\alpha} = 1-R^*/r \tag{5-28}$$

$$g'_{\alpha\beta} = \begin{pmatrix} -(1-R^*/r)^{-1} & & & \\ & -r^2 & & \\ & & -r^2\sin^2\theta & \\ & & & (1-R^*/r) \end{pmatrix} \tag{5-29}$$

$$g'^{\alpha\beta} = \begin{pmatrix} -(1-R^*/r) & & & \\ & -1/r^2 & & \\ & & -1/r^2\sin^2\theta & \\ & & & (1-R^*/r)^{-1} \end{pmatrix} . \tag{5-30}$$

We may now solve (3-2) for the scalar field Ψ. By (3-2) and (3-18)

$$g'^{\alpha\beta} \left[\Psi_{,\alpha\beta} - \left\{{\lambda \atop \alpha\beta}\right\}'\Psi_{,\lambda}\right] = 0 . \tag{5-31}$$

Since by (5-6) $\Psi = \gamma^{-1/3}$ and $\gamma = \gamma(r)$ we have $\Psi = \Psi(r)$ which gives immediately that $\Psi,_2 = \Psi,_3 = \Psi,_4 = 0$. Using this (5-31) becomes

$$g'^{11}\Psi,_{11} - g'^{\alpha\beta} \{^{1}_{\alpha\beta}\}'\Psi,_1 = 0 \; . \tag{5-32}$$

By (5-17) to (5-24) we see that the only non-vanishing Christoffel symbols of the type $\{^{1}_{\alpha\beta}\}'$ are $\{^{1}_{11}\}'$, $\{^{1}_{22}\}'$, $\{^{1}_{33}\}'$, $\{^{1}_{44}\}'$ and therefore (5-32) becomes

$$g'^{11}\Psi,_{11} - [g'^{11}\{^{1}_{11}\}' + g'^{22}\{^{1}_{22}\}' + g'^{33}\{^{1}_{33}\}' \tag{5-33}$$

$$+ g'^{44}\{^{1}_{44}\}']\Psi,_1 = 0 \; .$$

Substituting in the appropriate values from (5-17) to (5-20) and (5-12) to (5-15) we obtain

$$e^{-\alpha}\Psi,_{11} + \Big\{-e^{-\alpha}\,\frac{1}{2}\,d\alpha/dr + \frac{1}{r}\,e^{-\alpha} + \frac{1}{r}\,e^{-\alpha} \tag{5-34}$$

$$+ e^{-\beta}\,\frac{1}{2}\;\; e^{\beta-\alpha}\,d\beta/dr\Big\}\Psi,_1 = 0 \; .$$

By differentiating (5-28) with respect to r we obtain

$$e^{\beta}\,\frac{d\beta}{dr} = -e^{-\alpha}\,\frac{d\alpha}{dr} = \frac{R*}{r^2} \; . \tag{5-35}$$

Substituting this into (5-34) and using (5-28) there results

$$(1 - R*/r)\,\frac{d^2\Psi}{dr^2} + \Big\{\frac{1}{2}\,R*/r^2 + \frac{2}{r}\,(1 - R*/r) \tag{5-36}$$

$$+ \frac{1}{2}\,R*/r^2\Big\}\,\frac{d\Psi}{dr} = 0 \; .$$

A modicum of algebra reduces this to

$$\frac{d^2\Psi}{dr^2} + (r^2 - R^*r)^{-1}(2r - R^*)\frac{d\Psi}{dr} = 0 \ . \tag{5-37}$$

This equation can easily be integrated to obtain

$$\Psi = \frac{C_1}{R^*}\ln(1 - R^*/r) + C_2 \quad . \tag{5-38}$$

Finally by substituting (5-38) for Ψ and (5-29) for $g'_{\alpha\beta}$ into (5-5) we obtain

$$ds^2 = \left\{\exp\left[\frac{C_1}{R^*}\ln(1 - R^*/r) + C_2\right]^{-3}\right\} \tag{5-39}$$

$$\times \ \{-(1-R^*/r)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2$$

$$+ (1-R^*/r)c^2dt^2\} \ .$$

We can evaluate the constants C_1 and C_2 by appealing to our demand for MP compatibility. We note that as either $R^* \to 0$ or $r \to \infty$ the quantity $[\ln(1-R^*/r)]^{-3}$ approaches $-\infty$. Since a factor in the line element such as $e^{-\infty}$ will give us exactly what we want, we may take (other, less simple, choices are possible)

$$C_1 = \alpha R^* \ , \quad C_2 = 0 \tag{5-40}$$

where α is some positive dimensionless constant. In this case (5-39) becomes

$$ds^2 = e^{[\alpha \ln(1-R^*/r)]^{-3}} \ \{-(1-R^*/r)^{-1}dr^2 - r^2d\theta^2 \tag{5-41}$$

$$- r^2 \sin^2\theta d\varphi^2 + (1-R^*/r)c^2 dt^2\} .$$

We see that as $R^* \to 0$ or $r \to \infty$ we do indeed have the result $ds^2 \to 0$. Thus the line element (5-41) valid in the region exterior to a FSSMD possesses the desired MP compatibility. Moreover if $R \gg R^*$ as will be the case for all astronomical bodies except those in a very highly collapsed state we can make the approximation (recall that since $r > R$, if $R \gg R^*$ we also have $r \gg R^*$)

$$\ln\,(1-R^*/r) \approx -R^*/r \tag{5-42}$$

in which case (5-41) becomes

$$ds^2 = e^{-(r/\alpha R^*)^3} \{-(1-R^*/r)^{-1}\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + (1-R^*/r)c^2 dt^2\} . \tag{5-43}$$

Here the MP compatibility is manifest since it is easily seen that $ds^2 \to 0$ as $r \to \infty$ or $R^* \to 0$. Furthermore since the quantity αR^* determines the decay rate of the exponential, it is apparent that αR^* is in a very definite sense a measure of just how much space a given mass can endow with structure. Thus since R^* is essentially just the mass of a given body we get the interesting result that the more massive a body is, the more space for which it will provide structure. This seems to be an especially satisfactory dividend of this theory, for surely this feature is just what one would expect from Mach's ideas.

CHAPTER VI

INTERDEPENDENCE AND THE EQUATIONS OF MOTION

We now turn to the question of the equations of motion of the ST theory. We begin by giving a brief review of the subject of interdependence of the field equations and the equations of motion. That the equations of motion were contained within the field equations of GR was wholly unsuspected by Einstein in 1916. In fact, it was not until eleven years later that Einstein and Grommer published the first paper which looked at this problem. This interdependence is so exceptional that it is referred to by Graves as the "conceptual novelty of GR" (8).

The essential element of GR which effects this interdependence is its non-linearity. It can be shown that in any linear field theory one must always separately postulate the laws of motion (9). Thus non-linearity is a necessary condition for being able to derive the equations of motion. Whether or not it is also a sufficient condition is, however, still open to question. Bergmann once concluded that non-linearity was indeed sufficient (10). However, he failed to show how the derivation of the equations of motion might be carried out in the general case. Infeld, one of the leading workers in this area, has since stated that there is no general criterion for determining when the laws of motion can be derived from a non-linear field theory (11).

Since Einstein was originally not aware of the fact that the equations of motion were derivable from the field equations, he was forced to independently postulate the equations of motion, which he did by assuming that free particles (particles moving under the influence of gravity alone) travel along geodesics. This is known as the geodesic hypothesis. It is quite remarkable that in GR the DPEM, i.e. those equations of motion predicted by the field equations themselves, do indeed turn out to be just the metrical geodesics.

We have mentioned that non-linearity is the essential general feature of a field theory that brings about the interdependence of the field equations and the equations of motion. The particular feature that permits one to actually derive the DPEM is normally the existence of some mathematical (vector) identity based on the field equations. For instance in GR the field equations

$$R^{\alpha\beta} - \tfrac{1}{2} R \, g^{\alpha\beta} = - \varkappa \, T^{\alpha\beta} \qquad (6\text{-}1)$$

give rise to the equation

$$T^{\alpha\beta}{}_{;\beta} = 0 \qquad (6\text{-}2)$$

since the LHS is identically divergenceless. Using (6-2) it can be shown that the DPEM are just the metrical geodesics.

The ST theory also permits one to derive the DPEM, the derivation depending on the existence of the equation (3-28), viz.,

$$T^{\alpha\beta}{}_{;'\beta} = 0 \qquad (6\text{-}3)$$

which is the ST analogue of the GR result (6-2). However, as we will show, the DPEM turn out not to be the metrical geodesics, although in certain cases the difference is negligible.

We begin by reformulating (6-3) in terms of $g_{\alpha\beta}$. By definition

$$T^{\alpha\beta}{}_{;'\beta} = T^{\alpha\beta}{}_{,\beta} + \{{}^{\alpha}_{\mu\beta}\}' T^{\mu\beta} + \{{}^{\beta}_{\mu\beta}\}' T^{\alpha\mu} . \tag{6-4}$$

The quantities $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are related by

$$g_{\alpha\beta} = e^{\chi} g'_{\alpha\beta} \tag{6-5}$$

where

$$\chi = 1/\Psi^3 . \tag{6-6}$$

The relationship between the Christoffel symbols as computed for $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ is given by Synge (12) as

$$\{{}^{\mu}_{\alpha\beta}\}' = \{{}^{\mu}_{\alpha\beta}\} - A^{\mu}_{\alpha\beta} \tag{6-7}$$

where

$$A^{\mu}_{\alpha\beta} = \tfrac{1}{2} \left(\delta^{\mu}_{\alpha} \chi_{,\beta} + \delta^{\mu}_{\beta} \chi_{,\alpha} - g'_{\alpha\beta} g'^{\mu\lambda} \chi_{,\lambda} \right) . \tag{6-8}$$

(See Appendix A for a derivation of these and other formulas for conformally related space-times.) We note that the product $g'_{\alpha\beta} g'^{\mu\lambda}$ in (6-8) can be written as

$$g'_{\alpha\beta} g'^{\mu\lambda} = g_{\alpha\beta} g^{\mu\lambda} \tag{6-9}$$

since $g_{\alpha\beta} = e^{\chi} g'_{\alpha\beta}$ and $g^{\alpha\beta} = e^{-\chi} g'^{\alpha\beta}$. Substituting (6-7) and (6-8) into (6-4) and using (6-9) and the definition of $T^{\alpha\beta}{}_{;\beta}$ there results

$$T^{\alpha\beta}{}_{;'\beta} = T^{\alpha\beta}{}_{;\beta} - 3T^{\alpha\beta}\chi_{,\beta} + \tfrac{1}{2}\, g_{\mu\beta}\, g^{\alpha\lambda}\, \chi_{,\lambda}\, T^{\mu\beta} \; . \tag{6-10}$$

Using (6-3) and the fact that $g_{\mu\beta}T^{\mu\beta} = T$, (6-10) becomes

$$T^{\alpha\beta}{}_{;\beta} + \tfrac{1}{2}\,(Tg^{\alpha\beta} - 6T^{\alpha\beta})\,\chi_{,\beta} = 0 \quad . \tag{6-11}$$

Note that the second term here represents the difference between the ST result and the GR one, since the GR equation is simply $T^{\alpha\beta}{}_{;\beta} = 0$. Using the definition of $T^{\alpha\beta}{}_{;\beta}$ (6-11) becomes

$$T^{\alpha\beta}{}_{,\beta} + \left\{{\alpha \atop \lambda\beta}\right\}T^{\lambda\beta} + \left\{{\beta \atop \lambda\beta}\right\}T^{\alpha\lambda} + \tfrac{1}{2}\,(T\,g^{\alpha\beta} - 6\,T^{\alpha\beta})\,\chi_{,\beta} = 0 \; . \tag{6-12}$$

Finally, multiplying this equation through by $\sqrt{-g}$ and using the well-known identity

$$\left\{{\beta \atop \lambda\beta}\right\} = \frac{1}{\sqrt{-g}}\,(\sqrt{-g})_{,\lambda} \tag{6-13}$$

(see, for example, Spain (13)) we obtain

$$(\sqrt{-g}\;T^{\alpha\beta})_{,\beta} + \sqrt{-g}\;\left\{{\alpha \atop \lambda\beta}\right\}\,T^{\lambda\beta} + \tfrac{1}{2}\sqrt{-g}\;(T\,g^{\alpha\beta} - 6\,T^{\alpha\beta})\,\chi_{,\beta} = 0 \; . \tag{6-14}$$

We now investigate this equation following the method of Adler, Bazin and Schiffer (14). Consider therefore a very small globule of matter which as it moves in time will fill a small world tube, D_4, of space-time (see Figure 6-1). Within D_4 the energy tensor is given by

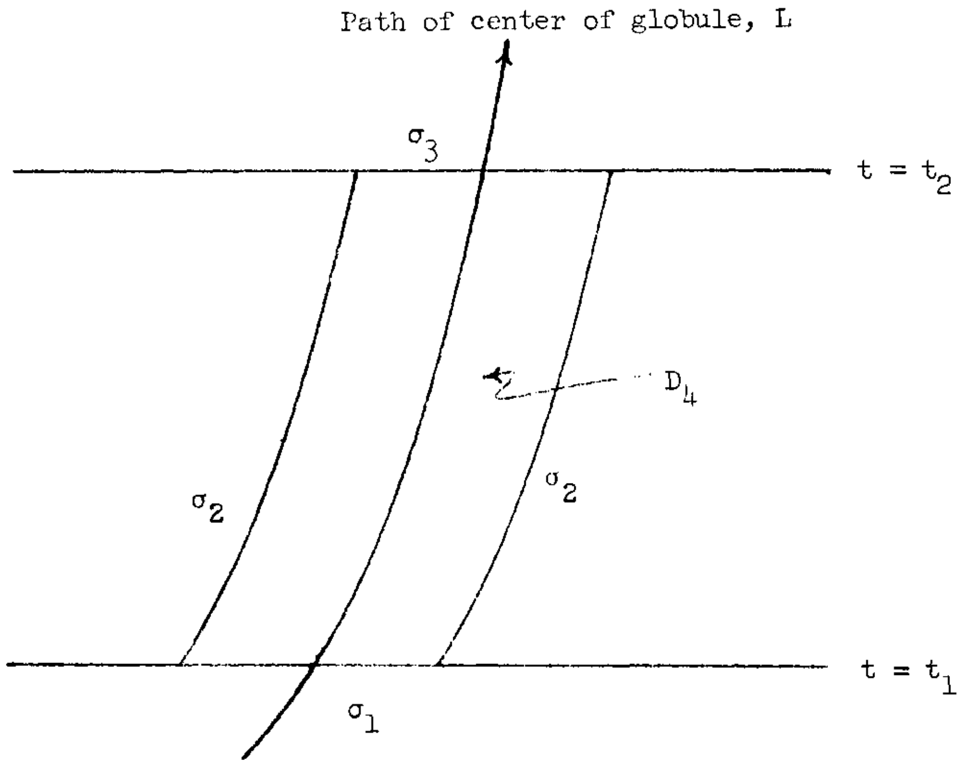

Figure 6-1. The World Tube of a Mass Globule

$$T^{\alpha\beta} = \rho_o \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \tag{6-15}$$

where $\rho_o(x)$ is the mass density of the globule and dx^α/ds is the four-velocity vector field. The density ρ_o is assumed to be different from zero only within the small volume of the globule and to vanish on the boundary. Also dx^α/ds is assumed to be nearly constant over this volume. In the course of its motion the globule will intersect the hypersurfaces $t=t_1$ and $t=t_2$ in the two small spatial regions σ_1 and σ_3 and will be bounded by the hypersurface σ_2. Eventually we will imagine the size of the globule to shrink down to zero in which case the world tube becomes the world line of a point particle. The limiting process is not, however, carried through with mathematical rigor.

Integrating (6-14) over the region D_4 we obtain

$$\int_{D_4} (\sqrt{-g}\, T^{\alpha\beta})_{,\beta}\, d^4x + \int_{D_4} [T^{\lambda\beta} \left\{{\alpha \atop \lambda\beta}\right\} + \tfrac{1}{2}(T\, g^{\alpha\beta} - 6T^{\alpha\beta})\, \chi_{,\beta}]\sqrt{-g}\, d^4x = 0 \quad . \tag{6-16}$$

Since the integrand of the first term is a complete divergence we can by Gauss's theorem write it as

$$I_1 = \int_{\sigma_1 + \sigma_2 + \sigma_3} \sqrt{-g}\, T^{\alpha\beta}\, n_\beta\, d\sigma \tag{6-17}$$

where $d\sigma$ is a three dimensional surface element and n_β is the normal to $d\sigma$. But $T^{\alpha\beta}$ vanishes on σ_2 and since the surface normals to σ_1 and σ_3 are, respectively, (0,0,0,-1) and (0,0,0,+1), (6-17) becomes

$$I_1 = \int_{\sigma_3} \sqrt{-g}\, T^{\alpha 4}\, d^3x - \int_{\sigma_1} \sqrt{-g}\, T^{\alpha 4}\, d^3x \tag{6-18}$$

where d^3x is the spatial volume element on the bounding hypersurfaces. Using (6-15) this can be written as

$$I_1 = \int_{\sigma_3} \rho_o \frac{dx^\alpha}{ds} \sqrt{-g} \frac{dx^4}{ds}\, d^3x - \int_{\sigma_1} \rho_o \frac{dx^\alpha}{ds} \sqrt{-g} \frac{dx^4}{ds}\, d^3x \; . \tag{6-19}$$

Next consider the term $\sqrt{-g}\, d^4x$. In Special Relativity this may be interpreted as the product of $c\, d\tau$ and dV where $d\tau$ and dV are, respectively, proper time and volume elements. Thus

$$\sqrt{-g}\, d^4x = c\, d\tau\, dV \; . \tag{6-20}$$

Furthermore we can use a locally Minkowskian coordinate system such that $ds = c d\tau$ so that

$$\sqrt{-g}\, d^4x = dV\, ds \; . \tag{6-21}$$

Finally then, since $d^4x = d^3x\, dx^4$ we have

$$\sqrt{-g}\, d^3x\, dx^4/ds = dV \; . \tag{6-22}$$

Substituting this into (6-19) we have

$$I_1 = \int_{\sigma_3} \rho_o \frac{dx^\alpha}{ds}\, dV - \int_{\sigma_1} \rho_o \frac{dx^\alpha}{ds}\, dV \; . \tag{6-23}$$

Since dx^α/ds was assumed to be nearly constant over the volume of the globule we can remove it from the integral to obtain

$$I_1 = m_o \, (dx^\alpha/ds)_{t=t_2} - m_o \, (dx^\alpha/ds)_{t=t_1} \tag{6-24}$$

where

$$m_o \equiv \int_{\sigma_1} \rho_o \, dV = \int_{\sigma_3} \rho_o \, dV \tag{6-25}$$

is the proper mass of the globule. If we now let the globule shrink down to a point particle of rest mass m_o we can write (6-24) as a line integral along L, the world line of the particle

$$I_1 = m_o \int_{L\,t_1}^{t_2} d^2x^\alpha/ds^2 \, ds \ . \tag{6-26}$$

Next consider the second term of (6-16). Using (6-15) and (6-21) we obtain

$$I_2 = \int_{D_4} \Big[\rho_o \frac{dx^\lambda}{ds}\frac{dx^\beta}{ds} \left\{{\alpha \atop \lambda\beta}\right\} + \tfrac{1}{2}\, \rho_o \, g^{\alpha\beta} \, \chi_{,\beta} \tag{6-27}$$

$$- 3 \, \rho_o \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \, \chi_{,\beta}\Big] \, ds \, dV$$

where we have used $T = \rho_o$. This follows from (6-15) since

$$T \equiv g_{\alpha\beta} \, T^{\alpha\beta} = g_{\alpha\beta} \, \rho_o \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = \rho_o \, \frac{g_{\alpha\beta} \, dx^\alpha \, dx^\beta}{ds^2} = \rho_o \ . \tag{6-28}$$

Factoring ρ_o out of the integrand in (6-27) and as before letting the volume become arbitrarily small we can perform the volume integration, obtaining

$$I_2 = \int_{L\,t_1}^{t_2} m_o \Big[\left\{ {\alpha \atop \lambda\beta} \right\} \frac{dx^\lambda}{ds}\frac{dx^\beta}{ds} + \tfrac{1}{2} g^{\alpha\beta} \chi_{,\beta}$$

$$- 3 \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \chi_{,\beta} \Big] ds \ . \qquad (6\text{-}29)$$

Combining (6-26) and (6-29) with (6-16) we get

$$m_o \int_{L\,t_1}^{t_2} \Big\{ \frac{d^2x^\alpha}{ds^2} + \left\{ {\alpha \atop \lambda\beta} \right\} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} + \tfrac{1}{2} g^{\alpha\beta} \chi_{,\beta}$$

$$- 3 \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \chi_{,\beta} \Big\} ds = 0 \ . \qquad (6\text{-}30)$$

Since the end points t_1 and t_2 are arbitrary, the integrand must vanish, and we obtain

$$\frac{d^2x^\alpha}{ds^2} + \left\{ {\alpha \atop \lambda\beta} \right\} \frac{dx^\lambda}{ds}\frac{dx^\beta}{ds} + \tfrac{1}{2}\left(g^{\alpha\beta} - 6 \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right) \chi_{,\beta} = 0 \ . \qquad (6\text{-}31)$$

Finally, from (6-6) we have

$$\chi_{,\beta} = \frac{-3}{\Psi^4} \Psi_{,\beta} \qquad (6\text{-}32)$$

and substituting this into (6-31) there results

$$\boxed{\frac{d^2x^\alpha}{ds^2} + \left\{ {\alpha \atop \lambda\beta} \right\} \frac{dx^\lambda}{ds}\frac{dx^\beta}{ds} - \frac{3}{2}\left(g^{\alpha\beta} - 6 \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right) \frac{\Psi_{,\beta}}{\Psi^4} = 0} \ . \qquad (6\text{-}33)$$

These are the DPEM of the ST theory. Note that the DPEM differ from the metrical geodesics by the presence of the third term only. Now if the scalar field Ψ is a very slowly varying function so that $\Psi_{,\beta} \approx 0$ then the DPEM closely approximate the metrical geodesics. We also note that as $\Psi \to \infty$ the third term disappears. This is in agreement with the analysis of Chapter IV where we showed that $\Psi \to \infty$ is the condition for the ST theory to reduce to GR.

These equations are not so formidable as they might at first appear and we will actually integrate them for the case of spherical symmetry in Chapter VIII.

We note that in the derivation we nowhere used the explicit form of the scalar equation (e.g. $\Box'\Psi = 0$) of the ST field equations. We used only $T^{\alpha\beta}{}_{;'\beta} = 0$ and $g_{\alpha\beta} = e^{\chi}\, g'_{\alpha\beta}$. Hence the equations (6-31) are the DPEM for any ST theory of the form

$$\left.\begin{array}{l} R'^{\alpha\beta} - \frac{1}{2} R'\, g'^{\alpha\beta} = - \varkappa\, T^{\alpha\beta} \\ \text{any scalar equation for } \chi \\ ds^2 = e^{\chi} g'_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} \end{array}\right\} \qquad (6\text{-}34)$$

This fact enables one to investigate other ST theories than the particular one proposed in this thesis.

CHAPTER VII

FIELD EQUATIONS IN TERMS OF $g_{\alpha\beta}$ AND Ψ

We show in this chapter how to express the ST field equations, (3-1) and (3-2), entirely in terms of $g_{\alpha\beta}$ and Ψ, that is, in a form in which $g'_{\alpha\beta}$ does not appear.

We consider the conformally related spaces $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ where

$$g_{\alpha\beta} = e^{\chi} g'_{\alpha\beta} \tag{7-1}$$

χ being an arbitrary function of the coordinates x^{α}. We make use of the equations (A-31) and (A-34) derived in Appendix A. These are

$$\begin{aligned} R'_{\alpha\beta} - \tfrac{1}{2} R' g'_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} R\, g_{\alpha\beta} - \chi_{;\alpha\beta} - \tfrac{1}{2} \chi_{,\alpha} \chi_{,\beta} \\ + g_{\alpha\beta}\left[\Box \chi - \tfrac{1}{4} \chi^{,\lambda} \chi_{,\lambda}\right] \end{aligned} \tag{7-2}$$

and

$$\Box' \chi = e^{\chi}(\Box \chi - \chi^{,\lambda} \chi_{,\lambda}) \tag{7-3}$$

where $\chi^{,\lambda} \chi_{,\lambda}$ is just the contraction of $\chi_{,\lambda}$, i.e. $\chi^{,\lambda} \chi_{,\lambda} = g^{\mu\lambda} \chi_{,\mu} \chi_{,\lambda}$. We recall that, following the convention adopted in Chapter III, all of the coderivatives on the RHS of (7-2) and (7-3) are computed with respect to $g_{\alpha\beta}$ since the relevant quantities are all unprimed. In the term $\Box'\chi$ on the LHS of (7-3) the coderivative must be taken with respect to $g'_{\alpha\beta}$ since it is a primed quantity.

Since the scalar field Ψ appears in the ST theory as

$$g_{\alpha\beta} = e^{1/\Psi^3} g'_{\alpha\beta} \tag{7-4}$$

we shall have to evaluate (7-2) and (7-3) for

$$\chi = 1/\Psi^3 \quad . \tag{7-5}$$

Thus we need to compute $(1/\Psi^3)_{,\alpha}$, $(1/\Psi^3)_{;\alpha\beta}$, $\Box(1/\Psi^3)$ and $(1/\Psi^3)^{,\lambda}(1/\Psi^3)_{,\lambda}$. Simple calculations show that

$$(1/\Psi^3)_{,\alpha} = -(3/\Psi^4)\,\Psi_{,\alpha} \tag{7-6}$$

$$(1/\Psi^3)_{;\alpha\beta} = -(3/\Psi^4)\,\Psi_{;\alpha\beta} + (12/\Psi^5)\,\Psi_{,\alpha}\,\Psi_{,\beta} \tag{7-7}$$

$$\Box\,(1/\Psi^3) = -(3/\Psi^4)\Box\Psi + (12/\Psi^5)\,\Psi^{,\lambda}\,\Psi_{,\lambda} \tag{7-8}$$

$$(1/\Psi^3)^{,\lambda}(1/\Psi^3)_{,\lambda} = (9/\Psi^8)\,\Psi_{,\lambda} \,. \tag{7-9}$$

Substituting (7-6) through (7-9) into (7-2) and recalling from (3-24) that $R'_{\alpha\beta} - \frac{1}{2} R'\, g'_{\alpha\beta} = -\varkappa\, e^{-2/\Psi^3}\, T_{\alpha\beta}$, we get

$$\boxed{\begin{aligned} &R_{\alpha\beta} - \tfrac{1}{2} R\, g_{\alpha\beta} - \left(\frac{12}{\Psi^5} + \frac{9/2}{\Psi^8}\right)\Psi_{,\alpha}\,\Psi_{,\beta} + \frac{3}{\Psi^4}\,\Psi_{;\alpha\beta} \\ &- \frac{3}{\Psi^4}\Box\Psi\, g_{\alpha\beta} + g_{\alpha\beta}\,\Psi^{,\lambda}\,\Psi_{,\lambda}\left(\frac{12}{\Psi^5} - \frac{9/4}{\Psi^8}\right) \\ &= -\varkappa\, e^{-2/\Psi^3}\, T_{\alpha\beta} \end{aligned}} \quad . \tag{7-10}$$

We next evaluate (7-3) for $\chi = 1/\Psi^3$.

$$\Box'(1/\Psi^3) = e^{1/\Psi^3}[\Box(1/\Psi^3) - (1/\Psi^3)^{,\lambda}(1/\Psi^3)_{,\lambda}] \quad . \tag{7-11}$$

From (7-8) we also have upon replacing $g_{\alpha\beta}$ everywhere with $g'_{\alpha\beta}$

$$\Box'(1/\Psi^3) = -(3/\Psi^4)\Box\Psi + (12/\Psi^5)\, g'^{\alpha\beta}\, \Psi_{,\alpha}\, \Psi_{,\beta} \; . \tag{7-12}$$

Equating (7-11) with (7-12) and recalling that the field equation (3-2) is just $\Box'\Psi = 0$ we obtain with the help of (7-8), (7-9) and (7-4)

$$\boxed{\Box\Psi + \frac{3}{\Psi^4}\, \Psi^{,\lambda}\, \Psi_{,\lambda} = 0} \quad . \tag{7-13}$$

The two boxed equations (7-10) and (7-13) represent the ST field equations expressed entirely in terms of $g_{\alpha\beta}$ and Ψ. Since the tensor field equation (7-10) appears rather unwieldy it is perhaps worthwhile to mention that the appearance of a large number of terms in the tensor equation is not an uncommon feature of ST theories. In fact, if a ST theory is derived from a variational principle there will normally be terms involving $\Psi_{;\alpha\beta}$, $\Psi_{,\alpha}\, \Psi_{,\beta}$, $g_{\alpha\beta}\Box\Psi$ and $g_{\alpha\beta}\, \Psi^{,\lambda}\, \Psi_{,\lambda}$, just as there are in (7-10). For example the Brans-Dicke theory (15) has the following tensor field equation:

$$R_{\alpha\beta} - \tfrac{1}{2} R\, g_{\alpha\beta} - \varphi^{-1}\, \varphi_{;\alpha\beta} - \omega\, \varphi^{-2}\, \varphi_{,\alpha}\, \varphi_{,\beta} \tag{7-14}$$

$$+ \varphi^{-1} g_{\alpha\beta} \Box\varphi + \tfrac{1}{2}\, \omega\, \varphi^{-2}\, g_{\alpha\beta}\, \varphi^{,\lambda}\, \varphi_{,\lambda}$$

$$= \frac{8\pi}{c^4}\, \varphi^{-1}\, T_{\alpha\beta}$$

where ω is a constant. Note the one to one correspondence between the six terms (the quantity $R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta}$ is considered to be a single quantity since it is just the Einstein tensor $G_{\alpha\beta}$) of (7-14) and those of (7-10).

Since (7-10) and (7-13) involve only the metric tensor and the scalar field Ψ these equations might seem preferable as the fundamental equations of the ST theory. However, for calculational purposes the simplication obtained by the mapping $g_{\alpha\beta} = e^{1/\Psi^3}\, g'_{\alpha\beta}$ is extremely useful. For then, of course, the equations become as before

$$\left.\begin{aligned} R'^{\alpha\beta} - \tfrac{1}{2} R'\, g'^{\alpha\beta} &= - \varkappa\, T^{\alpha\beta} \\ \Box' \Psi &= 0 \end{aligned}\right\} \quad . \qquad (7\text{-}15)$$

CHAPTER VIII

DIRECT PARTICLE EQUATIONS OF MOTION FOR FINITE SPHERICALLY SYMMETRIC MASS DISTRIBUTION

In this chapter we obtain the DPEM for a test particle moving in the region outside a FSSMD. We will show that these equations are very similar to the geodesics of the exterior Schwarzschild field in GR, and that under suitable conditions the two are indistinguishable. Thus we must solve the DPEM (6-33), viz.,

$$\frac{d^2x^\alpha}{ds^2} + \left\{{\alpha \atop \lambda\mu}\right\} \frac{dx^\lambda}{ds}\frac{dx^\mu}{ds} - \frac{3}{2}\left(g^{\alpha\beta} - 6\,\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)\frac{\Psi_{,\beta}}{\Psi^4} = 0 \tag{8-1}$$

in the region exterior to a FSSMD. The line element valid there is given by (5-41) as

$$ds^2 = e^{1/\Psi^3}\, g'_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{8-2}$$

where

$$\Psi = \alpha \ln(1 - R^*/r) \tag{8-3}$$

and

$$g'_{\alpha\beta} = \begin{pmatrix} -(1-R^*/r)^{-1} & & & \\ & -r^2 & & \\ & & -r^2\sin^2\theta & \\ & & & (1-R^*/r) \end{pmatrix} . \tag{8-4}$$

If we define

$$\chi = 1/\Psi^3 \tag{8-5}$$

and use a dot to denote d/ds, we can write (8-1) and (8-2), respectively, as

$$\ddot{x}^{\alpha} + \{^{\alpha}_{\lambda\mu}\}\,\dot{x}^{\lambda}\,\dot{x}^{\mu} + \tfrac{1}{2}\,(g^{\alpha\beta} - 6\,\dot{x}^{\alpha}\,\dot{x}^{\beta})\,\chi_{,\beta} = 0 \tag{8-6}$$

and

$$ds^2 = e^{\chi}\,g'_{\alpha\beta}\,dx^{\alpha}\,dx^{\beta} \quad . \tag{8-7}$$

We obtain a more usable form of (8-6) by multiplying the equation through by $g_{\alpha\tau}$ and using the definition of $\{^{\alpha}_{\lambda\mu}\}$. After combining like terms and changing some of the dummy indices there results

$$\frac{d}{ds}\,(g_{\alpha\beta}\,\dot{x}^{\beta}) - \tfrac{1}{2}\,g_{\lambda\beta,\alpha}\,\dot{x}^{\lambda}\,\dot{x}^{\beta} + \tfrac{1}{2}\,\chi_{,\alpha} - 3\,g_{\alpha\beta}\,\dot{x}^{\beta}\,\dot{x}^{\lambda}\,\chi_{,\lambda} = 0 \quad . \tag{8-8}$$

Making use of the fact that $g_{\alpha\beta} = e^{\chi}\,g'_{\alpha\beta}$, this becomes

$$e^{\chi}\,\frac{d}{ds}\,(g'_{\alpha\beta}\,\dot{x}^{\beta}) + g'_{\alpha\beta}\,\dot{x}^{\beta}\,e^{\chi}\,\dot{\chi} - \tfrac{1}{2}\,e^{\chi}\,g'_{\lambda\beta,\alpha}\,\dot{x}^{\lambda}\,\dot{x}^{\beta} - \tfrac{1}{2}\,g'_{\lambda\beta}\,e^{\chi}\,\chi_{,\alpha}\,\dot{x}^{\lambda}\,\dot{x}^{\beta} + \tfrac{1}{2}\,\chi_{,\alpha} - 3\,e^{\chi}\,g'_{\alpha\beta}\,\dot{x}^{\beta}\,\dot{x}^{\lambda}\,\chi_{,\lambda} = 0 \quad . \tag{8-9}$$

However, since $ds^2 = e^{\chi} g'_{\alpha\beta} \, dx^{\alpha} \, dx^{\beta}$ we have

$$e^{\chi} g'_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta} = \begin{cases} 1, & \text{particle motion} \\ 0, & \text{light motion} \end{cases} \quad . \tag{8-10}$$

(For light motion, the interval ds^2 is assumed, as in GR, to be null.) In the former case, the fourth term of (8-9) is just $-\frac{1}{2}\chi_{,\alpha}$ so that it cancels the fifth term. Upon dividing through by e^{χ} and noting that $\dot{x}^{\lambda} \chi_{,\lambda} \equiv \dot{\chi}$ we obtain

$$\frac{d}{ds}(g'_{\alpha\beta} \dot{x}^{\beta}) - \frac{1}{2} g'_{\lambda\beta,\alpha} \dot{x}^{\lambda} \dot{x}^{\beta} - 2 g'_{\alpha\beta} \dot{x}^{\beta} \dot{\chi} \tag{8-11}$$

$$= \begin{cases} 0, & \text{particle motion} \\ -\frac{1}{2} e^{-\chi} \chi_{,\alpha}, & \text{light motion} \end{cases} \quad .$$

This is the desired form of the DPEM since it permits an easy calculation of the differential equations of motion. In applying (8-11) to the case at hand we shall use the explicit form (8-4) for the $g'_{\alpha\beta}$. But for the moment we will let χ remain unspecified except for the restriction $\chi = \chi(r)$. We will, in fact, show that the DPEM can be integrated exactly, no matter what the actual functional form of χ. This feature of the ST theory is quite useful in that it enables one to readily investigate other ST theories of the particular class summarized by equations (6-34). Since $\chi = \chi(r)$ we have

$$\chi_{,2} = \chi_{,3} = \chi_{,4} = 0 \quad . \tag{8-12}$$

Finally we note that in solving the DPEM we may use either the four equations (8-11) or any three of these equations together with the first integral (8-10). We choose the latter course. Thus, using (8-4) and (8-12), (8-11) gives

$$\alpha=4: \qquad \frac{d}{ds}\left(g'_{44}\,\dot{x}^4\right) - 2\,g'_{44}\,\dot{x}^4\,\dot{\chi} = 0 \tag{8-13}$$

$$\alpha=3: \qquad \frac{d}{ds}\left(g'_{33}\,\dot{x}^3\right) - 2\,g'_{33}\,\dot{x}^3\,\dot{\chi} = 0 \tag{8-14}$$

$$\alpha=2: \qquad \frac{d}{ds}\left(g'_{22}\,\dot{x}^2\right) - \tfrac{1}{2}\,g'_{33,2}\,(\dot{x}^3)^2 - 2\,g'_{22}\,\dot{x}^2\,\dot{\chi} = 0 \tag{8-15}$$

and (8-10) becomes

$$g'_{11}\,(\dot{x}^1)^2 + g'_{22}\,(\dot{x}^2)^2 + g'_{33}\,(\dot{x}^3)^2 + g'_{44}\,(\dot{x}^4)^2 \tag{8-16}$$

$$= \begin{cases} e^{-\chi}, & \text{particle motion} \\ 0, & \text{light motion} \end{cases} \quad .$$

Defining

$$e^{\beta} = e^{-\alpha} = 1 - R^*/r \tag{8-17}$$

we can with the help of (8-4) write (8-13) through (8-16) as follows:

$$\frac{d}{ds}\left(e^{\beta}\,\dot{x}^4\right) - 2\,e^{\beta}\,\dot{x}^4\,\dot{\chi} = 0 \tag{8-18}$$

$$\frac{d}{ds}\left(r^2\,\sin^2\theta\,\dot{\varphi}\right) - 2\,r^2\,\sin^2\theta\,\dot{\varphi}\,\dot{\chi} = 0 \tag{8-19}$$

$$\frac{d}{ds}\left(r^2\,\theta\right) - \tfrac{1}{2}\left(2\,r^2\,\sin\theta\,\cos\theta\right)\dot{\varphi}^2 - 2\,r^2\,\dot{\theta}\,\dot{\chi} = 0 \tag{8-20}$$

$$-\,e^{\alpha}\,\dot{r}^2 - r^2\,\dot{\theta}^2 - r^2\,\sin^2\theta\,\dot{\varphi}^2 + e^{\beta}\,(\dot{x}^4)^2 \tag{8-21}$$

$$= \begin{cases} e^{-\chi}, & \text{particle motion} \\ 0, & \text{light motion} \end{cases} .$$

We assume the existence of plane solutions and take $\theta = \text{const} \equiv \pi/2$. Then (8-20) is identically satisfied and the remaining equations become

$$\left.\begin{aligned} &\frac{d}{ds}\left(e^{\beta}\dot{x}^4\right) - 2\left(e^{\beta}\dot{x}^4\right)\dot{\chi} = 0 \\ &\frac{d}{ds}\left(r^2\dot{\varphi}\right) - 2\left(r^2\dot{\varphi}\right)\dot{\chi} = 0 \\ &-e^{\alpha}\dot{r}^2 - r^2\dot{\varphi}^2 + e^{\beta}(\dot{x}^4)^2 = \begin{cases} e^{-\chi}, & \text{particle motion} \\ 0, & \text{light motion} \end{cases} \end{aligned}\right\} \quad (8\text{-}22) .$$

There are two features of these equations that are worthwhile noting at this juncture. Firstly, as $\chi \to 0$ (or equivalently as $\Psi \to \infty$) the above equations reduce to

$$\frac{d}{ds}\left(e^{\beta}\dot{x}^4\right) = 0$$

$$\frac{d}{ds}\left(r^2\dot{\varphi}\right) = 0 \qquad (8\text{-}23)$$

$$-e^{\alpha}\dot{r}^2 - r^2\dot{\varphi}^2 + e^{\beta}(\dot{x}^4)^2 = \begin{cases} 1, & \text{particle motion} \\ 0, & \text{light motion} \end{cases}$$

which are just the well-known geodesic equations of the exterior Schwarzschild field in GR. (See, for example, McVittie (16).) Secondly, it can be shown that the geodesic equations of motion (as opposed to the DPEM) for the same line element (8-7) have exactly the same form as (8-22) except that the coefficient -2 in the first two equations must be replaced by +1. This fact may be useful in examining other gravitational theories which predict a metric like (8-7) and which have the metrical geodesics as the DPEM.

Returning to the problem at hand we note that the first two equations of (8-22) have the form

$$\dot{z} - 2\, z\, \dot{\chi} = 0 \quad . \tag{8-24}$$

This has the solution

$$z = z_o\, e^{2\chi} \tag{8-25}$$

where z_o is a constant. Hence we can immediately write

$$e^{\beta}\, \dot{x}^4 = \gamma\, e^{2\chi} \tag{8-26}$$

$$r^2\, \dot{\varphi} = h\, e^{2\chi} \tag{8-27}$$

where γ and h are constants.

Particle Motion

We are now in a position to obtain the orbit equation for motion of a test particle about the central mass. Noting that $\dot{r} = \dot{\varphi}(dr/d\varphi)$ we can write the third equation of (8-22) as

$$- e^{\alpha}\, (dr/d\varphi)^2\, \dot{\varphi}^2 - r^2\, \dot{\varphi}^2 + e^{\beta}\, (\dot{x}^4)^2 = e^{-\chi} \quad . \tag{8-28}$$

Using (8-26) and (8-27) to eliminate the $\dot{x}^4$ and $\dot{\varphi}$ terms we obtain

$$- e^{\alpha} (dr/d\varphi)^2 - r^2 + \frac{\gamma^2}{h^2} r^4 e^{-\beta} = \frac{r^4}{h^2} e^{-5\chi} \quad . \tag{8-29}$$

The orbit equation is usually expressed in terms of u where

$$u \equiv 1/r \quad . \tag{8-30}$$

Defining u' as

$$u' = du/d\varphi \tag{8-31}$$

and using (8-17) there results after a modicum of algebra

$$u'^2 + u^2(1 - R^* u) - \gamma^2/h^2 \tag{8-32}$$

$$+ \frac{1}{h^2} (1 - R^* u) e^{-5\Sigma} = 0$$

where $\Sigma = \Sigma(u)$ is defined by

$$\Sigma(u) = \chi(1/u) \quad . \tag{8-33}$$

Finally, differentiation of (8-32) with respect to φ yields

$$u'' + u = \frac{R^*}{2h^2} e^{-5\Sigma} + \frac{3}{2} R^* u^2 + \frac{R^* u - 1}{2h^2} \frac{d}{du} (e^{-5\Sigma}) \tag{8-34}$$

which is the desired orbit equation.

This equation may be compared with the GR result which is usually written as

$$u'' + u = \frac{m}{h_o^2} + 3mu^2 \tag{8-35}$$

where

$$m = GM/c^2 \tag{8-36}$$

and

$$h_o = r^2 \dot{\varphi} = \text{const} \quad . \tag{8-37}$$

Loosely speaking, when (8-35) is applied to the solar system, the term m/h_o^2 gives rise to elliptical planetary orbits and the term $3mu^2$ causes these ellipses to precess. Thus we see from (8-34) that $e^{-5\Sigma}$ must be essentially equal to unity over the solar system in order for the ST result to agree with the well-known planetary motions. Furthermore we must also have $h = h_o$ and $R* = 2m$. Now the first of these conditions is automatically satisfied since by (8-37) and (8-27) h and h_o differ by the factor $e^{2\chi}(= e^{2\Sigma})$ which is unity whenever $e^{-5\Sigma}$ is unity.

The second condition can actually be interpreted as determining $R*$. This requires some discussion. We assumed in Chapter V that the constant of integration A of (5-27) could be shown to have the value $A = R* \equiv 2GM/c^2$. But this follows immediately upon requiring that the ST orbit equation reduce to the general relativistic one under suitable conditions. If $e^{-5\Sigma}$ is equal to 1 over the solar system, (8-34) becomes simply

$$u'' + u = \frac{R*}{2h_o^2} + \frac{3}{2} R* u^2 \tag{8-38}$$

where we have used the result discussed above that $h \to h_o$.

Comparison of this expression with (8-35) shows that $R^* = 2m$ and therefore by (8-36) that

$$R^* = 2GM/c^2 \quad . \tag{8-39}$$

Nor is this result very surprising. For although the ST theory differs from GR in both field equations and DPEM, the differences disappear as the scalar field $\Psi \to \infty$. But this is equivalent to $\Sigma \to 0$ which implies $e^{-5\Sigma} \to 1$ so that (8-34) should indeed reduce to the GR result as $e^{-5\Sigma} \to 1$.

We postpone (see Chapter XII) a detailed investigation of the extent to which it is legitimate to approximate $e^{-5\Sigma}$ by unity within the solar system until after we have looked at the quasar problem. For then we will have some idea as to the value of α, the numerical constant which appears in the solution of the ST field equations for the case of the FSSMD. (See (5-43).)

We note for future reference that the geodesics (rather than the DPEM) of (8-7) can be obtained from (8-34) by replacing -5Σ with Σ.

Light Motion

For light motion we must use the third equation of (8-22) in the form

$$- e^{\alpha}(dr/d\varphi)^2 \dot{\varphi}^2 - r^2 \dot{\varphi}^2 + e^{\beta} (\dot{x}^4)^2 = 0 \tag{8-40}$$

where again we have written $\dot{r}$ as $\dot{\varphi}(dr/d\varphi)$. If we divide this equation through by $\dot{\varphi}^2$ we will obtain an equation which is completely independent of χ. For by (8-26) and (8-27) the factors involving χ cancel out in the ratio $\dot{x}^4/\dot{\varphi}$. Thus the orbit equation for light motion can be obtained from (8-34) by eliminating the terms involving Σ (or equivalently χ). We get

$$u'' + u = \frac{3}{2} R^* u^2 \quad . \tag{8-41}$$

(Note that there are hand waving arguments by means of which this equation can be derived immediately. For instance, combining (8-27) with $ds^2 = 0$ implies that we must let either $h \to \infty$ or $\chi \to \infty$. Either of these conditions imposed on (8-34) will produce (8-41).)

Now (8-41) is just the GR result. Thus the scalar field will not affect the motion of light rays through space. Again this result is not surprising because of the way we have introduced the scalar field into the theory. For light will be bent if it undergoes a change of speed. But the speed is determined by the condition $ds^2 = 0$. However, $ds^2 = 0$ in conjunction with $ds^2 = e^{1/\Psi^3} g'_{\alpha\beta}\, dx^\alpha\, dx^\beta$ yields $g'_{\alpha\beta}\, dx^\alpha\, dx^\beta = 0$ so that only $g'_{\alpha\beta}$ affects the speed. Thus the same double bending of starlight that occurs in GR will necessarily occur in the ST theory.

There is an interesting analogy between this situation and the one which arises in the possible (purely tensor) gravitational theory

$$W_{\rho\sigma\lambda\tau} = 0 \tag{8-42}$$

where $W_{\rho\sigma\lambda\tau}$ is the Weyl tensor. It is well known that the vanishing of the Weyl tensor is a necessary and sufficient condition for the space-time to be conformally flat, i.e. for the line element to have the form

$$ds^2 = h(-\,dx^2 - dy^2 - dz^2 + c^2\,dt^2) \tag{8-43}$$

where h is an as yet unspecified function. h is, however, not arbitrary (17). This theory then gives rise to a gravitational field which will affect particles but not light rays (since of course, for $ds^2 = 0$, h disappears). Analogously the ST theory gives rise to a gravitational field in which both Ψ and $g'_{\alpha\beta}$ affect particle motions but only $g'_{\alpha\beta}$ affects light motion.

In summary of this chapter we may state that the double bending of starlight effect of GR will remain unchanged by the ST theory, and that under suitable conditions, to be discussed more fully later, the familiar planetary motions (including precessing of perihelia) still obtain. The affect of the ST theory on the third main prediction of GR, the gravitational red shift, forms the essence of Chapter X.

CHAPTER IX

OTHER POSSIBLE MACH'S PRINCIPLE COMPATIBLE THEORIES

We discuss here the question of whether there are other theories which might be compatible with MP as we have defined it in Chapter I. It is of essential importance in this regard to note that virtually any theory with the following three properties

1) the metric for the case of a FSSMD is conformally Schwarzschildian, i.e.

$$ds^2 = h(r,R^*)[-(1-R^*/r)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + (1-R^*/r)c^2dt^2] \quad (9\text{-}1)$$

2) $h(r,R^*) \to 0$ as $r \to \infty$ and $R^* \to 0$

3) the consequences of (9-1) for the problems of planetary motion, bending of starlight by the sun, and clock retardation, are indistinguishable from the consequences of the Schwarzschild metric in GR

is satisfactory. This is because the Schwarzschild metric is the only solution of curved space-time theories that has any present day experimental confirmation.

This thesis has so far been concerned with the ramifications of the ST theory:

$$R'^{\alpha\beta} - \tfrac{1}{2} R' g'^{\alpha\beta} = - \kappa T^{\alpha\beta} \quad (9\text{-}2)$$

$$\Box' \, \Psi = 0 \qquad (9\text{-}3)$$

$$h(\Psi) = e^{1/\Psi^3} \qquad (9\text{-}4)$$

$$ds^2 = h(\Psi) \; g'_{\alpha\beta} \; dx^\alpha \; dx^\beta \quad . \qquad (9\text{-}5)$$

We showed in Chapter V that properties (1) and (2) above are satisfied. In fact the conformally Schwarzschildian character of the FSSMD solution is automatically guaranteed by the form of (9-2) and (9-5) since for $T^{\alpha\beta} = 0$ and spherical symmetry the solution of (9-2) is the Schwarzschild metric. The fact that $h(\Psi)$ can be made to vanish as $r \to \infty$ or $R^* \to 0$ simply depends on a judicious choice of the field equation for Ψ and the functional h.

Scalar-tensor Theories

As far as other MP compatible ST theories are concerned we firstly note that we may replace (9-4) with

$$h(\Psi) = e^{-1/|\Psi|^n} \qquad (9\text{-}6)$$

for $n = 1,2,3,\ldots$ and still satisfy properties (1) and (2). For, as we have seen, the solution of (9-2) and (9-3) for a FSSMD can always be chosen so as to have the property $\Psi \to 0$ as $r \to \infty$ or $R^* \to 0$. But then $h(\Psi) \to 0$ as $r \to \infty$ or $R^* \to 0$ for any positive integer n. (We will not consider non-integer values of n.) In fact the choice $n = 3$ results from satisfying property (3). (See Chapter XII.)

Secondly we note that we may replace (9-2) with

$$R'^{\alpha\beta} = - \kappa \, T^{\alpha\beta} \qquad (9\text{-}7)$$

for whenever $T^{\alpha\beta} = 0$ (as it does in the region exterior to a FSSMD) the equations $R'^{\alpha\beta} - \frac{1}{2} R' \, g'^{\alpha\beta} = 0$ and $R'^{\alpha\beta} = 0$ are equivalent. Thus the conformally Schwarzschildian solution will still obtain for the case of a FSSMD and we can easily satisfy properties (1) and (2).

We now examine a theory of this type in more detail. Thus consider the equations

$$R'^{\alpha\beta} = - \varkappa \, T^{\alpha\beta} \tag{9-8}$$

$$\Box' \Psi = 0 \tag{9-9}$$

$$ds^2 = e^{1/\Psi^3} \, g'_{\alpha\beta} \, dx^\alpha \, dx^\beta \, . \tag{9-10}$$

Since this theory satisfies properties (1) and (2), it is MP compatible. But there is a significant difference between this theory and the one given by (9-2) to (9-5). For the LHS of (9-8) is no longer divergenceless, i.e. $R'^{\alpha\beta}{}_{;'\beta} \neq 0$ so that it is not necessary to require $T^{\alpha\beta}{}_{;'\beta} = 0$. But since the equations $T^{\alpha\beta}{}_{;'\beta} = 0$ determine the DPEM we no longer have the interdependence between field equations and equations of motion as discussed in Chapter VI. We are therefore free to independently postulate the equations of motion. The most natural choice is, of course, the geodesics. As Graves (18) states it is "probable that the geodesic law is compatible with other field equations based on somewhat different assumptions, but not clear whether it could also be derived from them. If not, the two laws would function as preconditions of each other." The theory (9-8) to (9-10) then becomes in full

$$\left.\begin{aligned} R'^{\alpha\beta} &= -\varkappa\, T^{\alpha\beta} \\ \Box'\Psi &= 0 \\ ds^2 &= e^{1/\Psi^3}\, g'_{\alpha\beta}\, dx^\alpha\, dx^\beta \\ \frac{d^2x^\alpha}{ds^2} &+ \left\{{\alpha \atop \lambda\beta}\right\} \frac{dx^\lambda}{ds}\frac{dx^\beta}{ds} = 0 \end{aligned}\right\} \qquad (9\text{-}11)$$

It might be argued that in losing the fundamental equation $T^{\alpha\beta}{}_{;\beta} = 0$ we no longer have the basic conservation laws of energy and momentum. However, the whole subject of conservation laws in relativistic field theories is somewhat ambiguous (as the following considerations will show) and it is probably too drastic to reject the theory (9-11) on this basis alone.

Einstein chose $R^{\alpha\beta} - \frac{1}{2} R\, g^{\alpha\beta}$ as the LHS of his field equations precisely in order that the "conservation law" $T^{\alpha\beta}{}_{;\beta} = 0$ follow as a consequence of the theory. However, as Graves (19) points out, this justification is both "unwarranted and false." It is unwarranted because conservation laws are based on ordinary, not covariant, divergences and false since $T^{\alpha\beta}$ by itself actually does not obey a conservation law.

What is needed is a set of quantities, $\Theta^{\alpha\beta}$, such that the ordinary divergence $\Theta^{\alpha\beta}{}_{,\beta} = 0$. For then the components $\Theta^{4\alpha}$ can be regarded as a measure of the energy and momentum densities. Such quantities can, in fact, be found within the framework of GR. The problem is that it is too easy to do so. To see this we write $T^{\beta}_{\alpha;\beta} = 0$ as

$$T^{\beta}_{\alpha;\beta} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, T^{\beta}_{\alpha}\right)_{,\beta} - \tfrac{1}{2} g_{\beta\rho,\alpha}\, T^{\beta\rho} = 0 \ . \tag{9-12}$$

Then if we can find quantities, t^{β}_{α}, such that

$$\frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, t^{\beta}_{\alpha}\right)_{,\beta} = - \tfrac{1}{2}\, g_{\beta\rho,\alpha}\, T^{\beta\rho} \tag{9-13}$$

we can write $\left(\sqrt{-g}\,\Theta^{\beta}_{\alpha}\right)_{,\beta} = 0$ where $\Theta^{\beta}_{\alpha} = T^{\beta}_{\alpha} + t^{\beta}_{\alpha}$. However, (9-13) provides us only with an equation for the divergence of t^{β}_{α}. Because of the mathematical property that the divergence of any curl is identically zero we can add an arbitrary curl term to any proposed t^{β}_{α} and still satisfy (9-13). Thus there are infinitely many ways to construct the Θ^{β}_{α}. Different choices have certain advantages in particular problems but none can claim universal validity. Because of this arbitariness in Θ^{β}_{α} it is not clear exactly what physical significance should be ascribed to these quantities.

In brief, it is not presently known precisely what the meaning of the "conservation law" $T^{\beta}_{\alpha;\beta} = 0$ is. This difficulty also manifests itself in the ST theory proposed in this thesis where the "conservation law" becomes $T^{\beta}_{\alpha;'\beta} = 0$. Using (6-10) we may express this in terms of $g_{\alpha\beta}$ and Ψ as

$$\frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, T^{\beta}_{\alpha}\right)_{,\beta} - \frac{1}{2}\, g_{\beta\rho,\alpha}\, T^{\beta\rho} - \frac{3}{2}\, T\, \frac{\Psi_{,\alpha}}{\Psi^{4}} + 9 T^{\beta}_{\alpha}\, \frac{\Psi_{,\beta}}{\Psi^{4}} = 0 \quad . \tag{9-14}$$

The previous analysis can be applied to this result provided we change (9-13) to include the extra terms present in (9-14) so that the equation for the t_{α}^{β} becomes

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\; t_{\alpha}^{\beta}\right)_{,\beta} = -\frac{1}{2} g_{\beta\rho,\alpha} T^{\beta\rho} - \frac{3}{2} T \frac{\Psi_{,\alpha}}{\Psi^4} + 9\, T_{\alpha}^{\beta} \frac{\Psi_{,\beta}}{\Psi^4} \qquad . \tag{9-15}$$

However, we don't presently know what the significance of the corresponding Θ_{α}^{β} has in regard to energy and momentum conservation. Further investigation of this matter is certainly warranted.

At any rate, aside from the questions of interdependence and conservation laws, much of the analysis of this thesis also pertains to the theory (9-11). This follows for two reasons. Firstly, the clock behavior and quasar analyses (Chapters X and XI) depend only on the form of the spherically symmetric line element, viz.

$$ds^2 = e^{-(r/\alpha R^*)^3}\left[\frac{-dr^2}{1-R^*/r} - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + (1-R^*/r)c^2 dt^2\right] \tag{9-16}$$

which also obtains in the theory (9-11). Secondly, as mentioned in Chapter VIII, the geodesic equations (recall that the DPEM of the theory (9-11) are the metrical geodesics) for motion about a FSSMD can be obtained from the DPEM of the ST theory by making the substitution $-5\Sigma \rightarrow \Sigma$ in (8-34). Therefore the remarks to be made in Chapter XII (which depend essentially on the smallness of Σ) as to the validity of the planetary orbit equation over the dimensions of the solar system (i.e. property (3)) will pertain to this theory as well.

The third major modification of the original ST theory that is worth considering involves simply inserting a T ($T = g_{\alpha\beta}\, T^{\alpha\beta}$) on the RHS of (9-3) so that

$$\Box' \Psi = T \quad . \tag{9-17}$$

Here a rather nice symmetry obtains in that matter now appears directly as a source for both $g'_{\alpha\beta}$ and Ψ whereas before Ψ depended only indirectly on the matter through its coupling with $g'_{\alpha\beta}$ in the operation $\Box' \Psi$. However, in the absence of matter, both $T^{\alpha\beta}$ and T vanish and, as in all of the theories so far examined, one obtains simply

$$\left.\begin{array}{l} R'^{\alpha\beta} = 0 \\ \Box' \Psi = 0 \end{array}\right\} \quad . \tag{9-18}$$

Eddington's Fourth Order Tensor Theories

Finally, we mention the possibility of achieving the desired results with a purely tensor theory. This could perhaps be accomplished by removing the restriction that the LHS of the general relativistic field equations (which are $R^{\alpha\beta} - \frac{1}{2} R\, g^{\alpha\beta} = - \varkappa\, T^{\alpha\beta}$) contain no derivatives higher than second order in $g_{\alpha\beta}$. In fact, Eddington (20) has partially investigated two of these higher order theories. The theories, both of which are fourth order in $g_{\alpha\beta}$, are expressed by the equations

$$\frac{\mathcal{H}}{\mathcal{H} g_{\alpha\beta}} \left(R_{\alpha\beta}\, R^{\alpha\beta}\right) = - \varkappa\, T^{\alpha\beta} \tag{9-19}$$

and

$$\frac{\mathcal{H}}{\mathcal{H} g_{\alpha\beta}} \left(R_{\alpha\beta\lambda\mu}\, R^{\alpha\beta\lambda\mu}\right) = - \varkappa\, T^{\alpha\beta} \tag{9-20}$$

where $\mathcal{H}/\mathcal{H}g_{\alpha\beta}$, the Hamiltonian derivative, is defined by

$$\delta \int K \sqrt{-g}\, d^4x = \int \frac{\mathcal{H} K}{\mathcal{H}g_{\alpha\beta}}\, \delta g_{\alpha\beta} \sqrt{-g}\, d^4x \quad . \tag{9-21}$$

(Note that the Einstein theory is expressible as $\mathcal{H}R/\mathcal{H}g_{\alpha\beta} = -\varkappa T^{\alpha\beta}$.) It turns out that in both (9-19) and (9-20), the LHS is divergenceless, so that in each case $T^{\alpha\beta}{}_{;\beta} = 0$. Both theories then have geodesics as the DPEM.

Eddington shows that any solution of Einstein's equations are also solutions of (9-19) and (9-20). But the converse is not necessarily true. He states "There are doubtless other . . . solutions for the alternative law of gravitation which are not permitted by Einstein's law, since the differential equations are now of the fourth order and involve two extra boundary conditions either at the particle or at infinity."

The outlook for the possibilities presented by this approach seems promising for the way we have formulated MP is precisely in the form of two boundary conditions, one at the particle and one at infinity.

PART III

CONSEQUENCES OF THE SCALAR-TENSOR THEORY

CHAPTER X

THE BEHAVIOR OF CLOCKS AND THE MACHIAN RED SHIFT

In Part II of this thesis we have set down a well defined ST theory of gravity and proved in detail that it satisfies a mathematical property which we believe is a reasonable interpretation of MP. In fact, the sole justification to this point for the seeming complication of the ST theory over GR lies in precisely this MP compatibility. In Part III we hope to show that the ST theory has much more to offer. It will be shown that some very definite physical effects are predicted, and that most importantly the possibility exists of explaining quasars as (cosmologically) local bodies.

We begin by considering the behavior of clocks in the region exterior to a spherically symmetric central body of mass M and radius R which is assumed to be alone in an otherwise empty universe. The central mass is further assumed to be a normal (i.e. uncollapsed) astronomical body so that $R \gg R^*$. Then we may utilize the solution obtained in Chapter V for the FSSMD problem in the form (5-43), viz.,

$$ds^2 = e^{-(r/\alpha R^*)^3} [-(1 - R^*/r)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 + (1 - R^*/r)c^2 dt^2] \, . \tag{10-1}$$

Let C_1 and C_2 be two fixed clocks located at P_1 and P_2, respectively (see Figure 10-1). Since the clocks are fixed in space dr, $d\theta$, and $d\varphi$ will vanish and we obtain

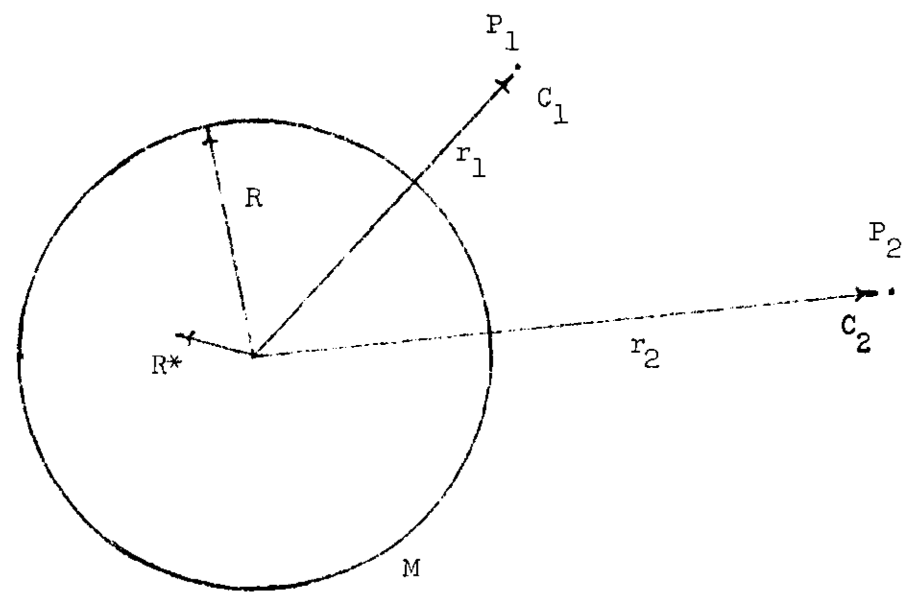

Figure 10-1. Clocks Outside a Finite Spherically Symmetric Mass Distribution

$$ds_1^2 = e^{-(r_1/\alpha R^*)^3} (1 - R^*/r_1)\ c^2 dt_1^2 \tag{10-2}$$

for C_1, and

$$ds_2^2 = e^{-(r_2/\alpha R^*)^3} (1 - R^*/r_2)\ c^2 dt_2^2 \tag{10-3}$$

for C_2.

For finite differences (10-2) and (10-3) become

$$\Delta s_1^2 = e^{-(r_1/\alpha R^*)^3} (1-R^*/r_1)c^2\ \Delta t_1^2 \tag{10-4}$$

$$\Delta s_2^2 = e^{-(r_2/\alpha R^*)^3} (1-R^*/r_2)c^2\ \Delta t_2^2\ . \tag{10-5}$$

Here we make use of Einstein's assumption of the equivalency of atomic clocks which states that

$$\Delta s_1 = \Delta s_2 \quad . \tag{10-6}$$

Physically this means that if two identical atoms, one at P_1 and the other at P_2, undergo the same atomic transition, the observed frequency must be the same when measured by the proper time. Thus from (10-4) and (10-5) after taking square roots we obtain

$$e^{-\frac{1}{2}(r_1/\alpha R^*)^3} (1-R^*/r_1)^{\frac{1}{2}}\ \Delta t_1 = e^{-\frac{1}{2}(r_2/\alpha R^*)^3} (1-R^*/r_2)^{\frac{1}{2}}\ \Delta t_2\ . \tag{10-7}$$

Letting P_1 correspond to the surface of M which gives r_1=R and solving (10-7) for Δt_2 yields

$$\Delta t = \frac{e^{-\frac{1}{2}(R/\alpha R^*)^3}\ (1-R^*/R)^{\frac{1}{2}}}{e^{-\frac{1}{2}(r/\alpha R^*)^3}\ (1-R^*/r)^{\frac{1}{2}}}\ \Delta T \qquad (10\text{-}8)$$

where we have made the notational changes $\Delta t_1 \rightarrow \Delta T$ and have dropped the subscript 2.

The quantity Δt, which is the coordinate period of a clock at an arbitrary distance r from the central body, is plotted in Figure 10-2 as a function of r. In constructing this graph we have assumed that $\alpha R^* \gg R$. Since $\alpha R^* \gg R$ the exponential term $\exp[-\frac{1}{2}(r/\alpha R^*)^3]$ is negligible for small r and

$$\Delta t \sim \frac{(1-R^*/R)^{\frac{1}{2}}}{(1-R^*/r)^{\frac{1}{2}}}\ \Delta T \qquad (10\text{-}9)$$

while for large r, $(1-R^*/r) \approx 1$ so that

$$\Delta t \sim e^{\frac{1}{2}(r/\alpha R^*)^3}\ \Delta t_{min} \qquad (10\text{-}10)$$

where

$$\Delta t_{min} \equiv (1-R^*/R)^{\frac{1}{2}}\ \Delta T \quad . \qquad (10\text{-}11)$$

(The term $\exp[-\frac{1}{2}(R/\alpha R^*)^3]$ is approximated by unity in all these formulas.)

Thus we see from (10-9) that a clock sitting on the surface of the mass M will beat at a rate ΔT. As the clock is moved radially away from M it will begin to speed up (since Δt decreases) and will asymptotically approach its maximum rate as r increases. However,

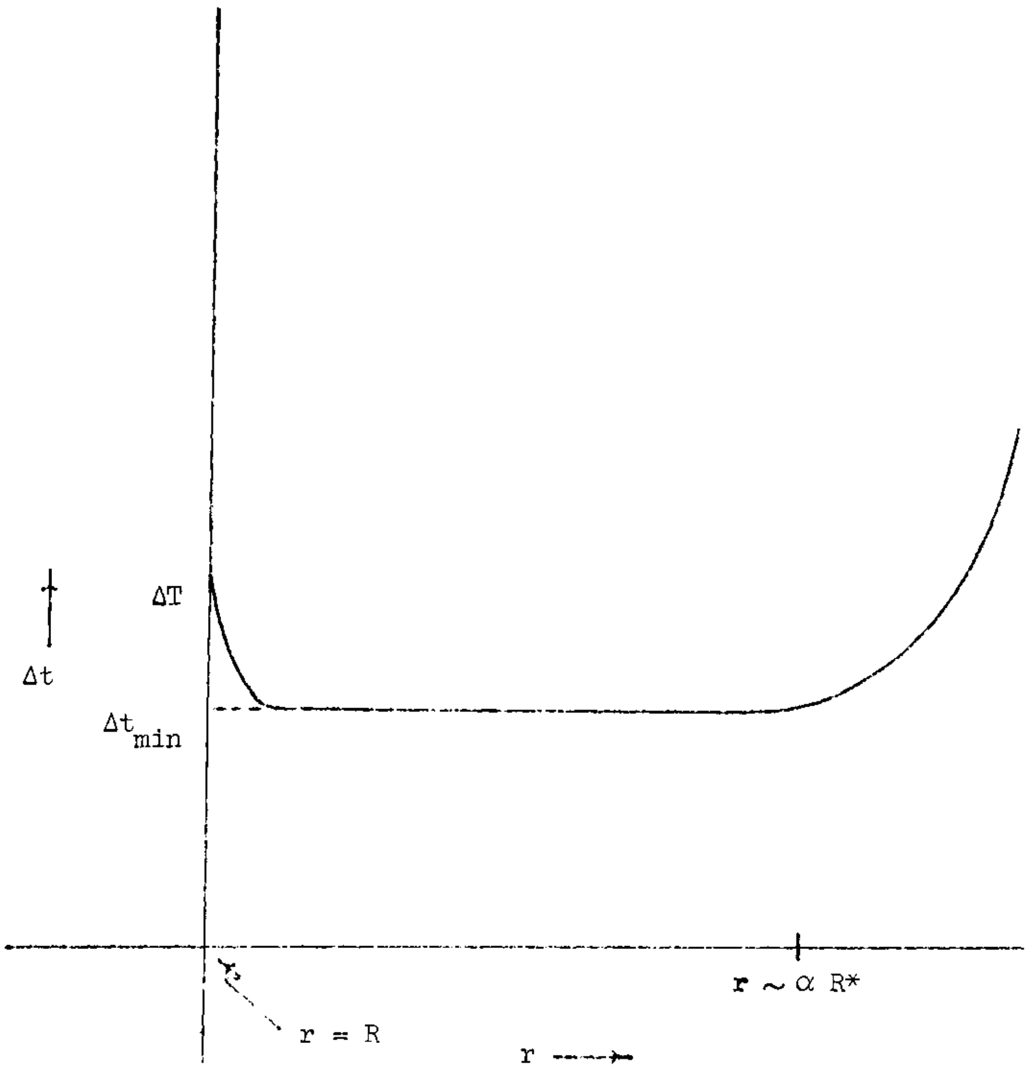

Figure 10-2. Clock Rate As Function of r

when distances of the order of $r = \alpha R^*$ are reached the exponential term in (10-10) causes Δt to increase and the clock will begin to slow down again. In fact as $r \to \infty$, $\Delta t \to \infty$ so that the clock will cease to tick at all at infinity. This is, of course to be expected from the MP compatibility of the line element (10-1). For as $r \to \infty$ space-time becomes structureless and clocks would be expected to stop.

We note that these results differ from GR only in the region $\alpha R^* \sim r < \infty$, where $\Delta t = \exp[\frac{1}{2}(r/\alpha R^*)^3]\ \Delta t_{min}$. In GR, Δt would remain at the value Δt_{min} as $r \to \infty$.

We now consider the implications of (10-8) from the standpoint of red shift (hereafter denoted RS). We consider that Δt is the period of a wave produced in some atomic transition. If λ and ν are the wave-length and frequency associated with this wave and λ_R, ν_R are similarly related to ΔT we can write (10-8) in the two equivalent forms

$$\lambda = \lambda_R \frac{(1-R^*/R)^{\frac{1}{2}}\ e^{-\frac{1}{2}(R/\alpha R^*)^3}}{(1-R^*/r)^{\frac{1}{2}}\ e^{-\frac{1}{2}(r/\alpha R^*)^3}} \tag{10-12}$$

$$\nu = \nu_R \frac{(1-R^*/r)^{\frac{1}{2}}\ e^{-\frac{1}{2}(r/\alpha R^*)^3}}{(1-R^*/R)^{\frac{1}{2}}\ e^{-\frac{1}{2}(R/\alpha R^*)^3}} \quad . \tag{10-13}$$

Since ΔT is associated with a clock on the surface of the mass M, λ_R and ν_R relate to waves produced at the surface.

Let us again suppose that $\alpha R^* \gg R$ so that in the region of small r, i.e. $R < r \ll \alpha R^*$, we may write (10-12) as

$$\lambda = \lambda_R \frac{(1-R^*/R)^{\frac{1}{2}}}{(1-R^*/r)^{\frac{1}{2}}} \quad . \tag{10-14}$$

This is just the familiar GR result which predicts the existence of the phenomenon known as the gravitational red shift. For by (10-14) λ_R is always greater than λ and consequently light produced at the surface of the massive body M will appear red shifted to an observer at the distance r. Equivalently if we are on the central body's surface and observe light coming to us from a small (for example a flashlight) external source, it will appear blue shifted.

Next consider light received by the central mass from a small source at distances r of the order of αR^*. Then the exponential term in (10-12) is not negligible and we must write

$$\lambda = \lambda_R \; e^{\frac{1}{2}(r/\alpha R^*)^3} \tag{10-15}$$

where because of the assumptions $\alpha R^* \gg R$ and $R \gg R^*$ the factors $(1-R^*/R)^{\frac{1}{2}}$, $(1-R^*/r)^{\frac{1}{2}}$, and $\exp[-\frac{1}{2}(R/\alpha R^*)^3]$ are all approximately unity.

We see from (10-15) that we must always have $\lambda > \lambda_R$. Thus the light received at the central body from the external source will be red shifted! Also, we note that because of the nature of the exponential function these red shifts can be quite large. Now this RS is completely opposite to the gravitational red shift of GR, since, as pointed out earlier, the gravitational red shift of GR actually leads to a blue shift for light moving towards a massive body. Furthermore, this new RS depends ultimately upon the requirement of MP compatibility since the exponential term of (10-15) arose originally (see (10-1)) in order to cause $ds^2 \to 0$ as $r \to \infty$ or $R^* \to 0$. For these reasons then, we propose to refer to this new RS as the Machian red shift and denote it by MRS. We will show in Chapter XI how the MRS can be used to explain quasars as cosmologically local bodies.

We emphasize that the MRS and the slowing down of clocks at distances of the order αR^* from the central body are, of course, equivalent effects. For light originating in the region $r \sim \alpha R^*$, where clocks are running slower than at $r = R$, will naturally have a comparatively longer wavelength.

Having looked at the relative rates of clocks in the vicinity of a given body we now turn to the question of the absolute magnitude of these rates. The rate ΔT of a fixed clock at the surface of the central mass M is from (10-1)

$$\Delta s = e^{-\frac{1}{2}(R/\alpha R^*)^3} (1-R^*/R)^{\frac{1}{2}} \Delta T \tag{10-16}$$

where we have set $c = 1$. In comparing the clock rates of two given bodies (i.e. the rates of clocks on the surface of the given bodies) we must equate their Δs values. Thus, for example, for an earth clock and a solar clock we obtain

$$e^{-\frac{1}{2}(R_E/\alpha R_E^*)^3} (1-R_E^*/R_E)^{\frac{1}{2}} \Delta T_E = e^{-\frac{1}{2}(R_S/\alpha R_S^*)^3} (1-R_S^*/R_S)^{\frac{1}{2}} \Delta T_S \ . \tag{10-17}$$

The relative rates therefore depend on the values of R^*/R and $R/\alpha R^*$ for the two bodies. For the earth

$$R_E^*/R_E \sim 10^{-9} \tag{10-18}$$

and for the sun

$$R_S^*/R_S \sim 10^{-6} \quad . \tag{10-19}$$

Assuming that $\alpha = 10^8$, (10-18) and (10-19) give

$$R_E/\alpha\ R_E^* \sim 10 \tag{10-20}$$

$$R_S/\alpha\ R_S^* \sim 10^{-2} \quad . \tag{10-21}$$

Using (10-18) through (10-21) and making the approximations $(1 - 10^{-9}) \approx 1$, $(1 - 10^{-6}) \approx 1$, and $\exp[-\frac{1}{2}(10^{-6})] \approx 1$, (10-17) becomes

$$\Delta T_E \sim \Delta T_S/e^{500} \quad . \tag{10-22}$$

Thus the earth clock would for all practical purposes not tick at all when compared to the solar clock. However, this result does not mean that this enormous rate difference actually occurs in nature. For it was tacitly assumed that each of the two bodies is completely independent. That is we have ignored both the rest of the matter in the universe and any mutual effect the earth and sun might have on each other's clocks. Most importantly we have ignored the galaxy.

We may consider the solar system as a point in the exterior field of the Milky Way core which we take to be a FSSMD. (See Figure 10-3.) Assuming that the core contains roughly half of the galactic matter, one obtains

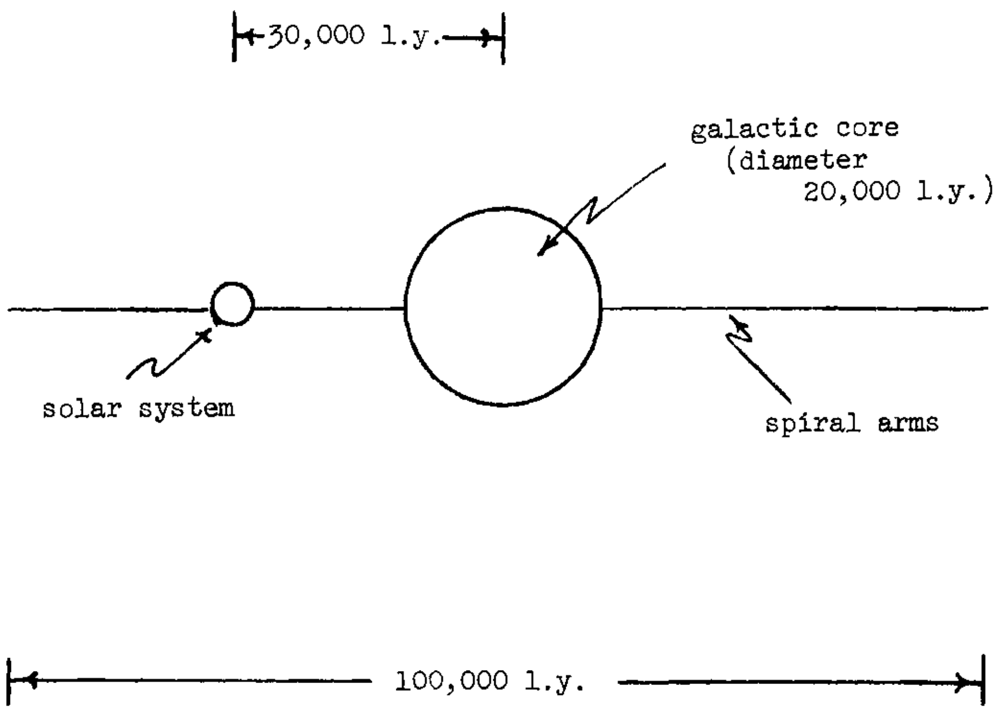

Figure 10-3. The Milky Way

$$R_G^*/R_G \sim 10^{-6} \tag{10-23}$$

which strikingly enough is the same order of magnitude as R_S^*/R_S. We have immediately from (10-16) that $\Delta T_G = \Delta T_S$. Thus the galaxy is capable of running clocks in its vicinity at rates equal to those obtained on normal stars such as the sun. This means that the galaxy will run the earth's clocks even though the earth itself is incapable of doing so!

The main idea we are trying to get across here is that clocks depend wholly on the presence of nearby matter for their ability to run. This is in sharp contrast to their behavior in GR where because of the Minkowski background clocks will always run at essentially their maximum allowable rate. In the ST theory clocks do not run at all when no matter is present, since the background space is structureless. As clocks are brought close to appreciable masses, their rates will increase drastically. Moreover, when a clock is in the presence of several gravitating bodies its ultimate rate will depend on the combined effect of all the masses involved. Thus, for example, the rate of earth clocks depends not only on the earth itself, but also on the sun and most importantly the galaxy. The sun's clock will basically be governed by the galaxy and the sun itself - the earth's effect being negligible because of its small mass. The fact that the earth clocks and sun clocks are observed in nature to run at essentially the same rate must be interpreted as due primarily to the influence of the galaxy.

CHAPTER XI

QUASARS AS COSMOLOGICALLY LOCAL BODIES

Companion Body Phenomenon

We are now in a position to understand how the MRS can be used to explain quasars as cosmologically local bodies. By way of introduction to this analysis we consider the astronomical configuration shown in Figure 11-1. Here G_1 and G_2 are two galaxies of roughly the same size, mass, and composition. C is a small companion body to G_1. The distance, D, separating the galaxies is assumed to be large compared to r_C, the distance between C and G_1.

We assume that C is of such a nature that it is incapable of running its own clocks. This will be true if $R_C^*/R_C < 10^{-9}$. For then with $\alpha = 10^8$, $R_C/\alpha R_C^* > 10$ and (10-16) gives

$$\Delta s_C < e^{-500} \Delta T_C \ . \qquad (11\text{-}1)$$

We also assume that C is luminous enough to be visible and distinct to G_2. We will shortly see that these conditions are easy to satisfy.

Although C cannot run its own clocks, the galaxy G_1 can. For we consider that C is well within the region for which G_1 generates structure. But here we have precisely the situation described in the previous chapter that led to the MRS provided that $r_C \sim \alpha R_{G_1}^*$. Thus if λ_C and λ_{G_1} are corresponding wavelengths produced at C and G_1 we have according to (10-15)

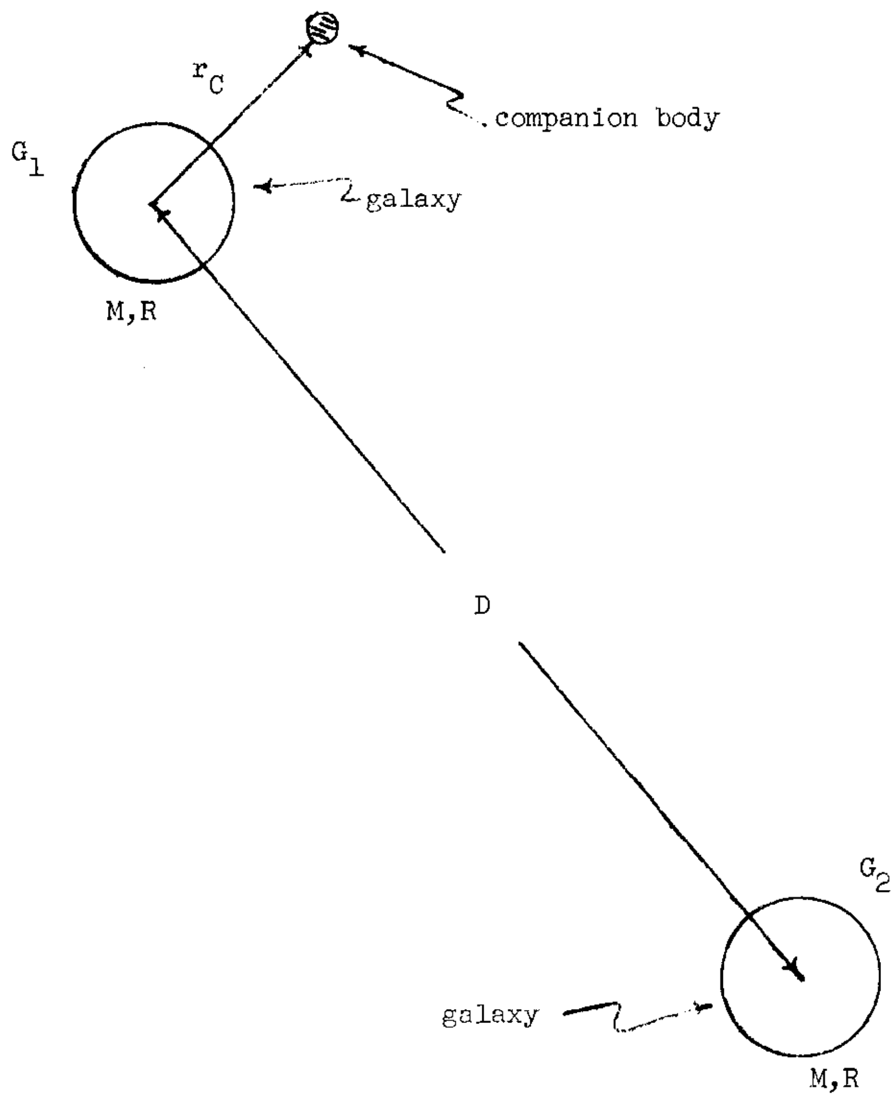

Figure 11-1. Companion Body to External Galaxy

$$\lambda_C = \lambda_{G_1} e^{\frac{1}{2}(r_C/\alpha R^*_{G_1})^3} \quad . \tag{11-2}$$

As before $\lambda_C > \lambda_{G_1}$ and light travelling from C to G_1 will be red shifted. And, as before, because of the exponential, this RS can be quite large.

Furthermore, whatever MRS obtains between C and G_1 will also obtain between C and G_2. This follows from our assumption of the equivalency of G_1 and G_2 which implies that the clocks of G_2 have the same rate as the clocks of G_1. Whatever difference there is between the rates of C's clocks and G_1's clocks will also be the difference between C's clocks and G_2's clocks. But, as we have pointed out earlier, the difference in clock rates between two bodies is equivalent to the difference in wavelengths. Thus we may replace λ_{G_1} in (11-2) by λ_{G_2} and obtain

$$\lambda_C = \lambda_{G_2} e^{\frac{1}{2}(r_C/\alpha R^*_{G_1})^3} \quad . \tag{11-3}$$

This is the desired result, for, light leaving C and arriving at G_2 will be red shifted. The magnitude of the RS will depend on the ratio $r_C/\alpha R^*_{G_1}$.

Quasars As Companion Bodies

Next we wish to show that it is possible to interpret quasars as precisely these companion bodies. Then G_2 corresponds to the Milky Way and G_1 to some nearby galaxy.

We briefly review the history of quasars (also known as quasi-stellar objects or QSO's). These strange bodies, discovered by Sandage in 1960, were originally detected because of their strong radio emission. They were then located optically and found to be starlike in appearance and to manifest very large red shifts. Not long after the original discovery, other starlike objects of large RS were found, which did not have strong radio emission. In fact, according to estimates by Burbidge and Hoyle (21) these silent quasi-stellars outnumber their radio counterparts by a factor of 100. Hence, the term quasar presently refers to <u>any</u> starlike object with a large RS, whether or not there is a significant radio output.

The quasar red shifts have been observed to fall in the range*

$$0.16 < z < 3 \tag{11-4}$$

although the upper limit increases almost daily. By way of contrast even the remotest galaxies have only $z \sim 0.46$. The usual interpretation of the quasar red shifts is that they are of a cosmological nature. That is, the quasars are extremely remote bodies, taking part in the overall expansion of the universe, whose recessional velocities are therefore governed by the Hubble law

$$v = H D \quad . \tag{11-5}$$

*The quantity z is known as the RS parameter and is defined by

$$z = (\lambda_Q - \lambda_E)/\lambda_E \quad .$$

Here λ_E is the laboratory wavelength corresponding to the spectral line emitted by the quasar.

Here, v is the recessional velocity, D the distance from our galaxy, and H Hubble's constant. The relationship between v and the RS parameter z is

$$\frac{v}{c} = \frac{(z + 1)^2 - 1}{(z + 1)^2 + 1} \quad . \tag{11-6}$$

(For small z, $v/c \approx z$.) In accepting this hypothesis, one is faced with the unpleasant task of explaining the enormous energy output (as much as several hundred times that of the brightest galaxies) of these starlike objects.

On the other hand, if we suppose that quasars are companion bodies to nearby galaxies, we can handle the large red shifts very easily by means of the MRS. At the same time, by making them local, we can reduce their required energy output to much less fantastic values.

Thus we suppose that the companion body C in Figure 11-1 is a quasar (denoted Q), that G_1 is a nearby galaxy (denoted G) and that G_2 is the Milky Way (denoted MW). With appropriate notational changes (11-3) becomes

$$\lambda_Q = \lambda_{MW}\, e^{\frac{1}{2}(r_Q/\alpha\, R_G^*)^3} \quad . \tag{11-7}$$

Since $z = \lambda_Q/\lambda_{MW} - 1$ we obtain after a modicum of algebra

$$r_Q = \alpha\, R_G^*(2 \ln (z + 1))^{1/3} \quad . \tag{11-8}$$

In order to accommodate the largest known RS of $z = 3$ we must have

$$r_Q = \alpha\ R^*_G (2\ \ln 4)^{1/3} \approx 1.4\ \alpha\ R^*_G \quad . \tag{11-9}$$

Thus, for example, if $\alpha = 10^8$ and $R^*_G/R_G \sim 10^{-7}$ we have $r_Q \sim 1.4\ R_G$. A quasar manifesting a smaller RS would be correspondingly closer to the companion galaxy.

Consistency of Quasar Data with Companion Body Analysis

We now examine the question of whether quasar data is consistent with this analysis. Primarily this involves determining whether or not the assumption $R_Q/\alpha\ R^*_Q > 10$ is legitimate. In order to get a handle on the value of R^*_Q/R_Q we proceed as follows. We write R^*_Q/R_Q as

$$\frac{R^*_Q}{R_Q} = \frac{R^*_Q}{R^*_S} \frac{R^*_S}{R_S} \frac{R_S}{R_Q} \tag{11-10}$$

where the subscript S stands for the sun. Since $R^* \propto M$ and $M \propto \rho R^3$ we have $R^* \propto \rho R^3$ and therefore we can write (11-10) as

$$\frac{R^*_Q}{R_Q} = \frac{\rho_Q}{\rho_S} \frac{R^*_S}{R_S} \left(\frac{R_Q}{R_S}\right)^2 \quad . \tag{11-11}$$

Next we express $(R_Q/R_S)^2$ in terms of the temperatures T_Q, T_S and absolute magnitudes M_Q, M_S. We assume that the energy source of quasars is thermonuclear. For then the Stefan-Boltzmann law is obeyed and the absolute luminosity L is proportional to $R^2\ T^4$, so that

$$\frac{L_Q}{L_S} = \frac{R_Q^2}{R_S^2} \cdot \frac{T_Q^4}{T_S^4} \quad . \tag{11-12}$$

The ratio L_Q/L_S is, however, defined in astronomy in terms of the absolute magnitudes as

$$\frac{L_Q}{L_S} = (100^{1/5})^{M_S - M_Q} \quad . \tag{11-13}$$

Eliminating L_Q/L_S from (11-12) and (11-13) and substituting the resulting expression for $(R_Q/R_S)^2$ into (11-11) there results

$$\frac{R_Q^*}{R_Q} = \frac{\rho_Q}{\rho_S} \frac{R_S^*}{R_S} \left(\frac{T_S}{T_Q}\right)^4 (100^{1/5})^{M_S - M_Q} \quad . \tag{11-14}$$

Using the luminosity distance relationship

$$M_Q = m_Q + 5 - 5 \log D_Q \tag{11-15}$$

where m_Q is the apparent magnitude of Q and D_Q is the distance in parsecs, we obtain

$$\frac{R_Q^*}{R_Q} = \frac{\rho_Q}{\rho_S} \frac{R_S^*}{R_S} \left(\frac{T_S}{T_Q}\right)^4 (100^{1/5})^{M_S - m_Q - 5 + 5 \log D_Q} \quad . \tag{11-16}$$

Taking $M_S \sim 5$ (actually $M_S = 4.8$) we can write this as

$$\frac{R_Q^*}{R_Q} = \frac{\rho_Q}{\rho_S} \frac{R_S^*}{R_S} \left(\frac{T_S}{T_Q}\right)^4 \frac{D_Q^2}{100^{m_Q/5}} \quad . \tag{11-17}$$

According to Burbidge and Hoyle (22) the strength of emission lines from the quasars indicates temperatures of the order of $T_Q \sim$ 30,000 ^oK and densities in the range $\rho_Q \sim 10^4$ to 10^7 particles/cm^3. Since the sun has $T_S \sim 6{,}000$ ^oK and $\rho_S \sim 10^{16}$ particles/cm^3 we have

$$T_Q/T_S \sim 5 \tag{11-18}$$

$$\rho_Q/\rho_S \sim 10^{-12} \text{ to } 10^{-9} \quad . \tag{11-19}$$

(The quasar spectrographs also indicate a rather normal starlike composition of elements.) The quasar apparent magnitudes (23) fall primarily in the range $m_Q \sim 15$ to 20. Combining this with (11-18), (11-19) and the fact that $R_S^*/R_S \sim 10^{-6}$, we have from (11-17)

$$\frac{R_Q^*}{R_Q} \sim (10^{-29} \text{ to } 10^{-24})D_Q^2 \quad . \tag{11-20}$$

Finally, the local hypothesis would demand that the quasars are located at distances of 1 to 100 Mpsc.* (1 Mpsc = 10^6 psc.) Then (since D_Q must be in psc) $D_Q \sim 10^6$ to 10^8 and (11-20) yields

$$R_Q^*/R_Q \sim 10^{-17} \text{ to } 10^{-8} \tag{11-21}$$

or,

$$R_Q/R_Q^* \sim 10^8 \text{ to } 10^{17} \quad . \tag{11-22}$$

*That 100 Mpsc is still local can be seen from the Hubble law. For since $H \sim 400$ km/sec/Mpsc, 100 Mpsc corresponds to a recessional speed of $v \sim 4 \times 10^4$ km/sec. This gives $z \approx \beta \sim .1$ which is still below the quasar RS range.

Since in arriving at this result we have carried through the full ranges of all the variables involved, it is quite reasonable to expect that all of the quasars will fall in the smaller range given by

$$R_Q/R_Q^* \sim 10^{10} \text{ to } 10^{15} \quad . \tag{11-23}$$

Then with $\alpha = 10^8$ we have succeeded in satisfying the condition $R_Q/\alpha\, R_Q^* > 10$ upon which the companion body analysis depends. In fact this is where the choice of $\alpha = 10^8$ comes from in the first place.

It is also of interest to compute the approximate sizes of the quasars. From (11-12), (11-13) and (11-15) we have

$$\left(\frac{R_Q}{R_S}\right)^2 = \left(\frac{T_S}{T_Q}\right)^4 (100^{1/5})^{M_S - m_Q - 5 + 5 \log D_Q} \quad . \tag{11-24}$$

Using (11-18) together with $M_S \sim 5$, $m_Q \sim 15$ to 20, and $D_Q \sim 10^6$ to 10^8, (11-24) gives

$$R_Q/R_S \sim 5 \text{ to } 5000 \quad . \tag{11-25}$$

The higher numbers in this range are of particular interest. For, the distance $R_Q \sim 5000\ R_S$ is roughly the size of the solar system which in turn is of the order of 1/5 light day. Now many of the quasars have been observed to vary in brightness (24) with periods as short as a day. The period of this pulsation τ_Q puts an upper limit of the order $c\ \tau_Q$ on the size of the quasar but no lower limit. Thus an observed period of $\tau_Q \sim 1$ day, corresponding to a maximum size of 1 light day, is not at all inconsistent with our determination of $R_Q \sim 1/5$ light day.

Furthermore, we note that the combination of large size and high temperature is just what is needed to make an object extremely luminous. For the luminosity increases with the square of the radius and with the fourth power of the temperature. Thus a quasar with $R_Q \sim 1000\ R_S$ and $T_Q \sim 5\ T_S$ would be roughly a billion times more luminous than the sun, even though no unusual sources of energy (such as gravitational collapse) have been postulated. These large and highly luminous bodies, are nevertheless still much smaller than galaxies and would appear starlike on photographic plates.

Thus we consider that the quasar data is, in fact, compatible with the various assumptions we have made. In summary we state that it is permissible to think of a quasar as a highly tenous, very hot and very large body, which depends completely on the presence of a companion galaxy (which is cosmologically local) to run its clocks. Its clocks will, however, run more slowly than those of the associated galaxy, a fact which necessarily leads to a MRS of its light when viewed from any other essentially equivalent galaxy. And it is precisely this MRS, not the cosmological one expressed by Hubble's law, that quasars manifest.

Additional Evidence for Companion Body Interpretation

Finally we mention that the recent efforts of many astronomers have indicated that there is considerable evidence for an association between galaxies and quasars. Halton Arp (25) in 1966 claimed observational evidence that the quasi-stellar objects are associated with nearby galaxies at distances of 10 to 100 Mpsc and suggested that the red shifts are due to some unknown mechanism. (This unknown mechanism is hopefully the MRS.) Later, in 1970, Arp (26) found by analyzing the distribution of quasars on the sky, that they statistically fall closer to the bright galaxies than the same number of randomly distributed objects would fall.

Still more recently a group working at La Jolla, California (Geoffrey and Margaret Burbidge, Phillip Solomon and P. A. Strittmatter) has claimed evidence that "lends further credibility to the idea that at least some of the QSO's are comparatively local objects, which are genetically related to the galaxies, and that their red shifts are not of cosmological origin." (27)

CHAPTER XII

GALAXY AS BACKGROUND OF SOLAR SYSTEM

We have seen in Chapter VIII that the orbit equation for the motion of a test particle about a FSSMD is

$$u'' + u = \frac{R_S^*}{2h^2} e^{-5\Sigma} + \frac{3}{2} R_S^* u^2 + \frac{R_S^* u - 1}{2h^2} \frac{d}{du} (e^{-5\Sigma}) \qquad (12\text{-}1)$$

where we now add the subscript S to R* to specifically indicate that the central body in question is the sun. This result was obtained from the DPEM for the line element

$$ds^2 = e^{\chi}[-(1-R_S^*/r)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2 \qquad (12\text{-}2)$$

$$+ (1-R_S^*/r)c^2 dt^2]$$

where Σ, χ and the scalar field Ψ are related by

$$\chi(r) = [1/\Psi(r)]^3 \qquad (12\text{-}3)$$

$$\Sigma(u) = \chi(1/u) \quad . \qquad (12\text{-}4)$$

The scalar field Ψ for this system was found in Chapter V (cf. (5-38)) to be

$$\Psi = \frac{C_1}{R_S^*} \ln (1 - R_S^*/r) \qquad (12\text{-}5)$$

where C_1 is a constant of integration. (The constant C_2 of (5-38) has been set equal to zero.) In satisfying our requirement of MP compatibility we were forced to take $C_1 = \alpha R^*_S$ so that $e^{1/\Psi^3} \to 0$ as $r \to \infty$ or $R^*_S \to 0$. For large r, (12-5) gives simply $\Psi = -C_1/r$ using (12-3), (12-2) becomes

$$ds^2 = e^{-(r/C_1)^3}[-(1-R^*_S/r)^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \qquad (12\text{-}6)$$

$$+ (1-R^*_S/r)c^2 dt^2] \quad .$$

Therefore the constant $C_1 = \alpha R^*_S$ determines how fast the metric will fall off to zero, i.e. how fast the space-time structure will disappear. This analysis depends, however, on the central body in question (here the sun) being alone in the universe. But, of course, in reality, the sun sits outside the core of the galaxy. Hence the rate at which the space-time structure actually disappears will be governed in the main not by the sun but by the galaxy. Thus it seems reasonable to choose $C_1 = \alpha R^*_G$ rather than $C_1 = \alpha R^*_S$. In this case (12-6) becomes

$$ds^2 = e^{-(r/\alpha R^*_G)^3}[-(1-R^*_S/r)^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \qquad (12\text{-}7)$$

$$+ (1-R^*_S/r)c^2 dt^2]$$

and it is this line element which we consider to be valid for the region outside the sun.

We wish to show that the exponential factor in (12-7) is negligible over the extent of the solar system or equivalently that (12-1) reduces to the GR result, viz.,

$$u'' + u = \frac{R_S^*}{2h^2} + \frac{3}{2} R_S^* u^2 \tag{12-8}$$

for all $r \leq D_{SS}$, where D_{SS} is the radius of the solar system.

Since we have taken $\Psi = - \alpha R_G^*/r$, (12-3) gives

$$\chi(r) = -(r/\alpha R_G^*)^3 \quad . \tag{12-9}$$

Then from (12-4)

$$\Sigma(u) = -1/(\alpha R_G^* u)^3 \quad . \tag{12-10}$$

With λ defined by

$$\lambda = 5^{1/3}/\alpha R_G^* \tag{12-11}$$

we have

$$e^{-5\Sigma} = e^{(\lambda/u)^3} \quad . \tag{12-12}$$

This gives

$$\frac{d}{du} (e^{-5\Sigma}) = - 3 \frac{\lambda^3}{u^4} e^{(\lambda/u)^3} \quad . \tag{12-13}$$

Now λ/u is quite small. For by definition $\lambda/u \sim r/\alpha R_G^*$, and since the largest values of r correspond to $r \sim D_{SS}$, we have $(\lambda/u)_{max} \sim D_{SS}/\alpha R_G^*$.

But $R_G^*/R_G \sim 10^{-6}$ and $\alpha = 10^8$ so that $\alpha R_G^* \sim 100\, R_G$ which gives $(\lambda/u)_{max} \sim D_{SS}/100\, R_G$. Since D_{SS} is very small compared to R_G we do indeed have $(\lambda/u)_{max} \ll 1$. Thus we may expand $\exp[(\lambda/u)^3]$ in (12-12) and (12-13) as

$$e^{(\lambda/u)^3} = 1 + (\lambda/u)^3 + \ldots \quad . \tag{12-14}$$

Substituting (12-12) and (12-13) into (12-1) and using (12-14) we obtain

$$u'' + u = \frac{R_S^*}{2h^2}\,[1 + (\lambda/u)^3 + \ldots] + \frac{3}{3} R_S^*\, u^2 \tag{12-15}$$

$$+ \frac{R_S^*}{2h^2}\,(u - 1/R_S^*)(-3\lambda^3/u^4)[1 + (\lambda/u)^3 + \ldots] \quad .$$

Upon factoring $R_S^*/2h^2$ out of the RHS and neglecting all terms of the order $(\lambda/u)^6$ or higher this becomes

$$u'' + u = \frac{R_S^*}{2h^2}\left[1 + 3h^2u^2 - 2\,\frac{\lambda^3}{u^3} + 3\,\frac{\lambda^3}{R_S^*\, u^4}\right] \quad . \tag{12-16}$$

The first two terms on the RHS of (12-16) are just the "ellipse" and "precession" terms of the GR orbit equation (12-8). Also as we will shortly show $h^2\, u^2 \ll 1$. Of the third and fourth terms on the RHS, the fourth is the larger since it essentially differs from the third by a factor of r/R_S^* which is much greater than one. Thus all we need to show is that the fourth term is negligible compared to $3h^2u^2$ throughout the solar system. Therefore we consider the ratio

$$\frac{3\lambda^3/R_S^* \, u^4}{3h^2 \, u^2} \quad . \tag{12-17}$$

Using (12-11) and the fact that $u = 1/r$ this becomes

$$\frac{5r^6}{h^2 \, R_S^* \, \alpha^3 \, R_G^{*3}} \quad . \tag{12-18}$$

By (8-27) $h \approx r^2 \, d\varphi/ds$ since $e^{2\chi} \approx 1$ over the solar system. Also $ds \approx c \, dt$ since the planetary speeds are much smaller than the speed of light. Hence $h \approx c^{-1} \, r^2 \, d\varphi/dt = r \, v_T/c$ where $v_T = r \, d\varphi/dt$ is the orbital velocity. Squaring this we obtain $h^2 = r^2 \, v_T^2/c^2$. (Note that this result justifies our earlier statement that $h^2 \, u^2 \ll 1$ since the planetary orbital speeds are much less than the speed of light.) Since the planetary orbits are approximately circular, $v_T^2/r \approx G \, M_S/r^2$, so that $v_T^2 \approx G \, M_S/r$. Thus $h^2 \approx r \, G \, M_S/c^2$. But $G \, M_S/c^2 \equiv \frac{1}{2} R_S^*$ so that, finally, $h^2 \approx r \, R_S^*/2$. Using this value for h^2 the ratio (12-18) becomes

$$\frac{10 \, r^5}{R_S^{*2} \, \alpha^3 \, R_G^{*3}} \quad . \tag{12-19}$$

This ratio will have its largest value when $r \sim D_{SS}$. Using this and recalling that $\alpha \, R_G^* \sim 100 \, R_G$ (12-19) becomes

$$10 \left(\frac{D_{SS}}{R_S^*}\right)^2 \left(\frac{D_{SS}}{100 \, R_G}\right)^3 \quad . \tag{12-20}$$

Using the values

$$D_{SS} \sim 6 \times 10^{9} \text{ km} \approx 6 \times 10^{-4} \text{ l.y.} \tag{12-21}$$

$$R_G \sim 10^{4} \text{ l.y.} \tag{12-22}$$

$$R_S^* \sim 3 \text{ km} \tag{12-23}$$

(12-20) becomes

$$10(2 \times 10^{9})^2 \, (6 \times 10^{-10})^3 \tag{12-24}$$

which quantity is indeed small, being of the order of 10^{-8}.

APPENDIX A

CONFORMAL TRANSFORMATIONS

We derive here some useful relationships between conformally related space-times. Consider then the two space-times $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ related by

$$g_{\alpha\beta} = e^{\chi}\, g'_{\alpha\beta} \tag{A-1}$$

where χ is an arbitrary function of the coordinates x^{α}. Obviously

$$g^{\alpha\beta} = e^{-\chi}\, g'^{\alpha\beta} \quad . \tag{A-2}$$

Since

$$\left\{{\lambda \atop \alpha\beta}\right\} = g^{\lambda\mu}[\alpha\beta,\mu] = g^{\lambda\mu}\tfrac{1}{2}(g_{\alpha\mu,\beta} + g_{\beta\mu,\alpha} - g_{\alpha\beta,\mu}) \tag{A-3}$$

we have using (A-1) and (A-2)

$$\left\{{\lambda \atop \alpha\beta}\right\} = e^{-\chi}\, g'^{\lambda\mu}\, \tfrac{1}{2}[(e^{\chi}g'_{\alpha\mu})_{,\beta} + (e^{\chi}g'_{\beta\mu})_{,\alpha} - (e^{\chi}g'_{\alpha\beta})_{,\mu}] \quad . \tag{A-4}$$

This gives

$$\left\{{\lambda \atop \alpha\beta}\right\} = g'^{\lambda\mu}\, \tfrac{1}{2}(g'_{\alpha\mu,\beta} + g'_{\beta\mu,\alpha} - g'_{\alpha\beta,\mu}) \tag{A-5}$$

$$+ g'^{\lambda\mu}\, \tfrac{1}{2}(g'_{\alpha\mu}\chi_{,\beta} + g'_{\beta\mu}\chi_{,\alpha} - g'_{\alpha\beta}\chi_{,\mu})$$

or

$$\left\{{\lambda \atop \alpha\beta}\right\} = \left\{{\lambda \atop \alpha\beta}\right\}' + A^{\lambda}_{\alpha\beta} \tag{A-6}$$

where

$$A^{\lambda}_{\alpha\beta} \equiv \tfrac{1}{2}(\delta^{\lambda}_{\alpha}\ \chi_{,\beta} + \delta^{\lambda}_{\beta}\ \chi_{,\alpha} - g'^{\lambda\mu} g'_{\alpha\beta}\ \chi_{,\mu}) \quad . \tag{A-7}$$

Also by (A-1) and (A-2)

$$g'^{\lambda\mu} g'_{\alpha\beta} = e^{\chi}\ e^{\lambda\mu}\ e^{-\chi}\ g_{\alpha\beta} = g^{\lambda\mu}\ g_{\alpha\beta} \tag{A-8}$$

so that we can write $A^{\lambda}_{\alpha\beta}$ as

$$A^{\lambda}_{\alpha\beta} = \tfrac{1}{2}(\delta^{\lambda}_{\alpha}\ \chi_{,\beta} + \delta^{\lambda}_{\beta}\ \chi_{,\alpha} - g_{\alpha\beta}\ g^{\lambda\mu}\ \chi_{,\mu}) \quad . \tag{A-9}$$

The Ricci tensor for $g'_{\alpha\beta}$ is (cf. (3-5))

$$R'_{\alpha\beta} = - \left\{{\lambda \atop \alpha\beta}\right\}'_{,\lambda} + \left\{{\lambda \atop \alpha\lambda}\right\}'_{,\beta} + \left\{{\lambda \atop \mu\beta}\right\}' \left\{{\mu \atop \alpha\lambda}\right\}' - \left\{{\lambda \atop \alpha\beta}\right\}' \left\{{\mu \atop \lambda\mu}\right\}' \quad . \tag{A-10}$$

Substituting (A-6) into this expression there results

$$\begin{aligned} R'_{\alpha\beta} = &- \left[\left\{{\lambda \atop \alpha\beta}\right\} - A^{\lambda}_{\alpha\beta}\right]_{,\lambda} + \left[\left\{{\lambda \atop \alpha\lambda}\right\} - A^{\lambda}_{\alpha\lambda}\right]_{,\beta} \\ &+ \left[\left\{{\lambda \atop \mu\beta}\right\} - A^{\lambda}_{\mu\beta}\right]\left[\left\{{\mu \atop \alpha\lambda}\right\} - A^{\mu}_{\alpha\lambda}\right] - \left[\left\{{\lambda \atop \alpha\beta}\right\} - A^{\lambda}_{\alpha\beta}\right]\left[\left\{{\mu \atop \lambda\mu}\right\} - A^{\mu}_{\lambda\mu}\right] \end{aligned} \tag{A-11}$$

where it is evident that with $A^{\lambda}_{\alpha\beta}$ given by (A-9) the RHS is expressed entirely in terms of $g_{\alpha\beta}$ and χ. Writing (A-11) out we obtain

$$\begin{aligned} R'_{\alpha\beta} = &- \left\{{\lambda \atop \alpha\beta}\right\}_{,\lambda} + \left\{{\lambda \atop \alpha\lambda}\right\}_{,\beta} + \left\{{\lambda \atop \mu\beta}\right\}\left\{{\mu \atop \alpha\lambda}\right\} - \left\{{\lambda \atop \alpha\beta}\right\}\left\{{\mu \atop \lambda\mu}\right\} - A^{\lambda}_{\alpha\lambda,\beta} \\ &+ \left\{{\lambda \atop \alpha\beta}\right\} A^{\mu}_{\lambda\mu} - \left\{{\lambda \atop \mu\beta}\right\} A^{\mu}_{\alpha\lambda} + \left\{{\mu \atop \beta\lambda}\right\} A^{\lambda}_{\alpha\mu} + A^{\lambda}_{\alpha\beta,\lambda} \end{aligned} \tag{A-12}$$

$$+ \left\{{\mu \atop \lambda\mu}\right\} A^{\lambda}_{\alpha\beta} - \left\{{\mu \atop \alpha\lambda}\right\} A^{\lambda}_{\mu\beta} - \left\{{\mu \atop \beta\lambda}\right\} A^{\lambda}_{\alpha\mu} + A^{\lambda}_{\mu\beta} A^{\mu}_{\alpha\lambda} - A^{\lambda}_{\alpha\beta} A^{\mu}_{\lambda\mu}$$

where on the RHS we have added and subtracted the term $\left\{{\mu \atop \beta\lambda}\right\} A^{\lambda}_{\alpha\mu}$. By inspection we see that the first four terms are just $R_{\alpha\beta}$, the second four terms are equal to $- A^{\lambda}_{\alpha\lambda;\beta}$, and the third four terms are equal to $A^{\lambda}_{\alpha\beta;\lambda}$. Thus

$$R'_{\alpha\beta} = R_{\alpha\beta} - A^{\lambda}_{\alpha\lambda;\beta} + A^{\lambda}_{\alpha\beta;\lambda} + A^{\lambda}_{\mu\beta} A^{\mu}_{\alpha\lambda} - A^{\lambda}_{\alpha\beta} A^{\mu}_{\lambda\mu} \quad . \tag{A-13}$$

Here, of course, following the notational convention introduced in Chapter III the coderivatives on the RHS are to be computed with respect to $g_{\alpha\beta}$.

We can evaluate the terms in (A-13) involving $A^{\lambda}_{\alpha\beta}$ by using (A-9). $A^{\lambda}_{\alpha\lambda}$ becomes

$$A^{\lambda}_{\alpha\lambda} = \tfrac{1}{2}(\delta^{\lambda}_{\alpha}\, \chi_{,\lambda} + \delta^{\lambda}_{\lambda}\, \chi_{,\alpha} - g_{\alpha\lambda}\, g^{\lambda\mu}\, \chi_{,\mu}) \tag{A-14}$$

or

$$A^{\lambda}_{\alpha\lambda} = \tfrac{1}{2}(\chi_{,\alpha} + 4\chi_{,\alpha} - \chi_{,\alpha}) \tag{A-15}$$

and

$$A^{\lambda}_{\alpha\lambda} = 2\chi_{,\alpha} \quad . \tag{A-16}$$

This last result gives immediately that

$$A^{\lambda}_{\alpha\lambda;\beta} = 2\chi_{;\alpha\beta} \quad . \tag{A-17}$$

Next we have from (A-9) that

$$A^{\lambda}_{\alpha\beta;\lambda} = \tfrac{1}{2}(\delta^{\lambda}_{\alpha}\,\chi_{;\beta\lambda} + \delta^{\lambda}_{\beta}\,\chi_{;\alpha\lambda} - g_{\alpha\beta}\,g^{\lambda\mu}\,\chi_{;\mu\lambda}) \tag{A-18}$$

where we have made use of the fact that the metric tensor has a vanishing coderivative. Since $\chi_{;\alpha\beta} = \chi_{;\beta\alpha}$ (in general, codifferentiation is not commutative) (A-18) becomes

$$A^{\lambda}_{\alpha\beta;\lambda} = \chi_{;\alpha\beta} - \tfrac{1}{2}\,g_{\alpha\beta}\,\Box\,\chi \tag{A-19}$$

where

$$\Box\;\chi \equiv g^{\alpha\beta}\;\chi_{;\alpha\beta} \quad . \tag{A-20}$$

Next we have

$$A^{\lambda}_{\mu\beta}\,A^{\mu}_{\alpha\lambda} = \tfrac{1}{2}(\delta^{\lambda}_{\mu}\,\chi_{,\beta} + \delta^{\lambda}_{\beta}\,\chi_{,\mu} - g_{\mu\beta}\,g^{\lambda\tau}\,\chi_{,\tau}) \tag{A-21}$$

$$\times\ \tfrac{1}{2}(\delta^{\mu}_{\alpha}\,\chi_{,\lambda} + \delta^{\mu}_{\lambda}\,\chi_{,\alpha} - g_{\alpha\lambda}\,g^{\mu\sigma}\,\chi_{,\sigma})$$

which upon expanding the RHS becomes

$$A^{\lambda}_{\mu\beta}\,A^{\mu}_{\alpha\lambda} = \tfrac{1}{4}\,[\delta^{\lambda}_{\alpha}\,\chi_{,\lambda}\,\chi_{,\beta} + \delta^{\lambda}_{\lambda}\,\chi_{,\alpha}\,\chi_{,\beta} - \delta^{\sigma}_{\alpha}\,\chi_{,\beta}\,\chi_{,\sigma} + \chi_{,\alpha}\,\chi_{,\beta} \tag{A-22}$$

$$+\ \delta^{\mu}_{\beta}\,\chi_{,\mu}\,\chi_{,\alpha} - g_{\alpha\beta}\,g^{\mu\sigma}\,\chi_{,\sigma}\,\chi_{,\mu} - g_{\alpha\beta}\,g^{\lambda\tau}\,\chi_{,\tau}\,\chi_{,\lambda}$$

$$-\ \delta^{\tau}_{\beta}\,\chi_{,\tau}\,\chi_{,\alpha} + \delta^{\tau}_{\alpha}\delta^{\sigma}_{\beta}\,\chi_{,\tau}\,\chi_{,\sigma}] \quad .$$

Upon collecting like terms this becomes

$$A^{\lambda}_{\mu\beta}\, A^{\mu}_{\alpha\lambda} = \frac{3}{2}\, \chi_{,\alpha}\, \chi_{,\beta} - \tfrac{1}{2} g_{\alpha\beta}\, \chi^{,\lambda}\, \chi_{,\lambda} \tag{A-23}$$

where

$$\chi^{,\lambda}\, \chi_{,\lambda} \equiv g^{\lambda\mu}\, \chi_{,\lambda}\, \chi_{,\mu} \quad . \tag{A-24}$$

Next using (A-16) we obtain

$$A^{\lambda}_{\alpha\beta}\, A^{\mu}_{\lambda\mu} = \tfrac{1}{2}(\delta^{\lambda}_{\alpha}\, \chi_{,\beta} + \delta^{\lambda}_{\beta}\, \chi_{,\alpha} - g_{\alpha\beta}\, g^{\lambda\tau}\, \chi_{,\tau}) 2\chi_{,\lambda} \tag{A-25}$$

which gives

$$A^{\lambda}_{\alpha\beta}\, A^{\mu}_{\lambda\mu} = 2\chi_{,\alpha}\, \chi_{,\beta} - g_{\alpha\beta}\, \chi^{,\lambda}\, \chi_{,\lambda} \quad . \tag{A-26}$$

Finally substituting (A-17), (A-19), (A-23), and (A-26) into (A-13) we obtain

$$\begin{aligned} R'_{\alpha\beta} = R_{\alpha\beta} &- 2\chi_{;\alpha\beta} + \chi_{;\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} \Box \chi + \frac{3}{2}\, \chi_{,\alpha}\, \chi_{,\beta} + \\ &- \tfrac{1}{2} g_{\alpha\beta}\, \chi^{,\lambda}\, \chi_{,\lambda} - 2\chi_{,\alpha}\, \chi_{,\beta} + g_{\alpha\beta}\, \chi^{,\lambda}\, \chi_{,\lambda} \end{aligned} \tag{A-27}$$

so that

$$R'_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} \Box \chi + \tfrac{1}{2} g_{\alpha\beta} \chi^{,\lambda}\, \chi_{,\lambda} - \chi_{;\alpha\beta} - \tfrac{1}{2}\chi_{,\alpha}\, \chi_{,\beta} \tag{A-28}$$

which is the relationship between the Ricci tensors of $g'_{\alpha\beta}$ and $g_{\alpha\beta}$. Multiplying (A-28) by $g'^{\alpha\beta}$ there results

$$R' = g'^{\alpha\beta}\, R'_{\alpha\beta} = e^{\chi}\, g^{\alpha\beta}\, R'_{\alpha\beta} \tag{A-29}$$

$$= e^{\chi}[R - 2\Box\chi + 2\chi^{,\lambda}\chi_{,\lambda} - \Box\chi - \tfrac{1}{2}\chi^{,\lambda}\chi_{,\lambda}]$$

and

$$R' = e^{\chi}(R - 3\Box\chi + \tfrac{3}{2}\chi^{,\lambda}\chi_{,\lambda}) \quad . \tag{A-30}$$

Combining (A-28) and (A-30) and using (A-1) we obtain the relationship between the Einstein tensors

$$R'_{\alpha\beta} - \tfrac{1}{2} R' g'_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} + g_{\alpha\beta}\Box\chi \tag{A-31}$$

$$- \tfrac{1}{4} g_{\alpha\beta}\chi^{,\lambda}\chi_{,\lambda} - \chi_{;\alpha\beta} - \tfrac{1}{2}\chi_{,\alpha}\chi_{,\beta} \quad .$$

Lastly we derive the relationship between $\Box\chi$ and $\Box'\chi$. By definition

$$\Box'\chi = g'^{\alpha\beta}[\chi_{,\alpha\beta} - \{^{\lambda}_{\alpha\beta}\}'\chi_{,\lambda}] \tag{A-32}$$

and using (A-2), (A-6) and (A-9) this expression becomes

$$\Box'\chi = e^{\chi} g^{\alpha\beta}[\chi_{,\alpha\beta} - (\{^{\lambda}_{\alpha\beta}\} - \tfrac{1}{2}(\delta^{\lambda}_{\alpha}\chi_{,\beta} + \delta^{\lambda}_{\beta}\chi_{,\alpha} \tag{A-33}$$

$$- g_{\alpha\beta} g^{\lambda\mu}\chi_{,\mu}))\chi_{,\lambda}] \quad .$$

This can easily be seen to reduce to

$$\Box'\chi = e^{\chi}(\Box\chi - \chi^{,\lambda}\chi_{,\lambda}) \quad . \tag{A-34}$$

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